Connections and Symmetries in Spacetime

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The extent to which a symmetric, metric connection on spacetime determines the metric is given, and some applications to affine collineations are discussed.

1. INTRODUCTION

The object of this paper is to show to what extent the (symmetric) connection on a spacetime manifold $M$ determines the Lorentz metric on $M$. A corollary on the existence of proper affine collineations (affine collineations which are not homotheties or isometries) of $M$ is deduced and some further consequences of this result are discussed. Throughout the paper $M$ denotes a connected, simply connected (and hence time-orientable) spacetime manifold. All connections considered are symmetric and a connection $\Gamma$ on $M$ is called \textit{metric} if it is compatible with some Lorentz metric on $M$. All structures considered are assumed smooth, although this condition can often be weakened.

2. CONNECTIONS IN SPACETIME

Let $g$ be a Lorentz metric on $M$, let $p \in M$, and denote by $T_p M$ the tangent space to $M$ at $p$. The Lorentz metric $g_p$ at $p$ determines a pseudo-orthogonal group isomorphic to the full Lorentz group $\mathcal{L}$ which acts on $T_p M$ as a Lie transformation group in a natural way. Now suppose $\Gamma$ is a connection on $M$. Then $\Gamma$ gives rise at each $p \in M$ to the holonomy group $\Phi_p$ which, since $M$ is simply connected, is a connected Lie group and may

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be regarded as a group of linear transformations of $T_pM$. Since $M$ is connected (and hence path-connected), $\Phi_p$ and $\Phi_q$ are isomorphic for any $p, q \in M$. If $\Gamma$ is a metric connection compatible with $g$, then $\Phi_p$ is a connected subgroup of the identity component of the pseudo-orthogonal group at $p$ determined by $g_p$. Thus $\Phi_p$ uniquely determines a Lie subalgebra $A$ (assumed throughout to be nontrivial)\(^2\) of the Lorentz algebra $L$ (the holonomy algebra) and is then, equal to one of the 14 nontrivial types of Lie subalgebras of $L$ (which, with one exception to be discussed later, are listed in Table I) and is, of course, isomorphic to the Lie algebra determined at any other point $q \in M$ by $\Phi_q$. The listing in Table I uses the representation of $A$ as bivectors and is described in terms of an appropriate tetrad at a given point $p \in M$. All the subgroups arising from the algebras in Table I, except $R_{15}$, are reducible in the sense that the corresponding action on $T_pM$ leaves some subspaces of $T_pM$ invariant.

If the holonomy algebra is of type $R_9$, $R_{12}$, or $R_{14}$, a one-dimensional distribution (in the sense of Fröbenius) is determined at each $p \in M$. This distribution is null and describes the invariant subspaces of the holonomy group at each $p \in M$. It is thus smooth and, of course, integrable (for a general discussion of holonomy group theory see \([1]\)) and gives rise to a global nowhere-zero null vector field on $M$ by insisting that the local vector fields determined by the distribution each have unit inner product with a fixed global nowhere-zero timelike vector field which necessarily exists since $(M, g)$ is time-orientable. The resulting null vector field $l$ is then recurrent ($\nabla l = \mu l$, where $\nabla$ is the covariant derivative operator from $\Gamma$ and $\mu$ is a real-valued function on $M$) on account of the holonomy.

If the holonomy algebra is of the form $R_6$, $R_8$, $R_{10}$, $R_{11}$, or $R_{13}$, one again finds a one-dimensional distribution of invariant subspaces uniquely determined as above. For $R_{13}$ the distribution is timelike and the local vector fields which arise may be chosen to be future-pointing and of unit norm and give rise to a global timelike vector field $u$ on $M$. The vector field $u$ is then covariantly constant, $\nabla u = 0$. For $R_6$ or $R_{10}$, a spacelike one-dimensional distribution is uniquely determined on $M$ and the local vector fields which arise, when scaled so as to have unit norm, determine a “vector field up to sign” on $M$. The sign may be chosen consistently on $M$ by noting that, if one could not so choose it, one could construct a double covering manifold of $M$ in a well-known way, thus contradicting the fact that $M$ is simply connected. The resulting global spacelike vector field is then covariantly constant. If the holonomy algebra is of the form $R_8$ or $R_{11}$,

\(^2\)If $A$ is trivial the simply connectedness of $M$ implies that $\phi_p$ is trivial. Then $M$ is locally Minkowski space and the metric $g$ is uniquely determined up to an additive covariantly constant second-order symmetric tensor on $M$ (but preserving the Lorentz signature).