Homothetic Transformations with Fixed Points in Spacetime

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A study is made of homothetic motions with fixed points in spacetime. Some general properties of such spacetimes are established, and a characterization of generalized plane wave spacetimes is proved. A general discussion of homothetic motions in Einstein's theory is given.

1. INTRODUCTION

There has been some recent interest in the study of homothetic transformations in general relativity. This paper is intended as a contribution to this study and is concerned with those spacetimes which admit (proper, that is, nonisometric) homothetic transformations with fixed points. Since the preparation of this paper began, the author has discovered that similar results have been given independently by Alexeevski [1]. However, the proofs given here are shorter and in a form more accessible to relativists.

Let \((M, g)\) be a nonflat spacetime\(^2\) with smooth Lorentz metric \(g\) on \(M\). All structures on \(M\) will be supposed smooth. Suppose \(M\) admits a group (or local group) of homothetic transformations (or local homothetic transformations) of \(M\) which is generated by a vector field \(\xi\) on \(M\). Then \(\xi\) generates a local one-parameter group of local transformations \(\phi_t\) (\(t\) in some open interval about 0 in \(\mathbb{R}\)) in a neighborhood of any \(m \in M\) and the scaling of \(\xi\) may be assumed chosen so that \(\phi_t^*g = e^{-2t}g\) which, in Lie derivative notation, gives \(\mathcal{L}_{\xi} g = 2g\). In local coordinates one has \(\xi_{a;b} = \ldots\)

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\(^2\) This is in the sense that no open subset of \(M\) is flat.
\( g_{ab} + F_{ab} \), where a semicolon denotes a covariant derivative and \( F_{ab} = -F_{ba} \) is the homothetic bivector. It follows that if \( K = \xi^a \xi^a, \) \( K_{a\xi^a} = 2K \) where a comma denotes a partial derivative, and so \( K \) is an eigenfunction of \( \xi \) with nonzero eigenvalue [2]. As a consequence, the nature (timelike, spacelike, or null) of \( \xi \) is constant along any integral curve of \( \xi \), periodic nonnull integral curves of \( \xi \) are forbidden, and homothetic vector fields cannot exist on compact spacetimes.\(^3\) The vector field \( \xi \) must also satisfy

\[
\xi_{a;bc} = F_{ab;c} = R_{abcd} \xi^d
\]

where \( R \) is the curvature tensor. From this it follows that \( F_{ab}, F^{ab} \) (where the asterisk is the usual duality operator), \( A = F_{ab} F^{ab} \) and \( B = F_{ab} F^{ab} \) are covariantly constant along the integral curves of \( \xi \) (a consequence of the relations \( F_{ab;c} \xi^c = 0 \), etc.). Hence, if one classifies nonzero bivectors in the usual way as either nonsimple (\( B \neq 0 \)) or, if simple, (\( B = 0 \)), as either timelike (\( A < 0 \)), spacelike (\( A > 0 \)), or null (\( A = 0 \)), the type of the bivector \( F \) is constant along any integral curve of \( \xi \). The homothetic nature of \( \xi \) also implies that the curvature, Ricci, and Weyl tensors satisfy the following relations (with the usual abuse of notation)

\[
\mathcal{L}_\xi R_{abcd} = 0, \quad \mathcal{L}_\xi R_{ab} = 0, \quad \mathcal{L}_\xi C_{abcd} = 0
\]

Let \( p \in M, L \) the (range of the) integral curve of \( \xi \) through \( p \), and \( \phi_t(p) = q \in L \). If \( \tau^q_p \) denotes parallel translation \( T_q M \to T_p M \) between the tangent spaces to \( M \) at \( q \) and \( p \) along \( L \), then, since each \( \phi_t \) is affine, the set \( \{ C_t = \tau^q_p \circ \phi_{t\ast}, T_p M \to T_p M \} \) is a local one-parameter group of linear transformations of \( T_p M \) [3, Vol. I, p. 245] and \( C_t = \exp(tD) \), where \( D \) is the matrix \( \delta^a_b + F^a_b \) evaluated at \( p \). A geometrical interpretation of the homothetic bivector \( F \) is obtained by selecting a tetrad at \( p \) and propagating it along \( L \) by parallel transport. The action of \( \phi_{t\ast} \) with respect to this tetrad is then described exponentially by the matrix \( \delta^a_b + F^a_b \) at \( p \) and leads to the usual scaling factor together with a Lorentz transformation represented by \( F \) in a natural way. This Lorentz transformation is then a rotation, a boost, a proper null rotation, or a screw motion according as \( F \) is spacelike, timelike, null, or nonsimple at \( p \).

2. FIXED POINTS OF HOMOTHEITIES

Suppose in the above discussion that \( p \in M \) is a fixed point of the mappings \( \phi_t \) so that \( \xi(p) = 0 \). Then \( C_t = \phi_{t\ast} = \exp[t(\delta^a_b + F^a_b)] = e^t \exp(tF) \) is

\(^3\) However, \( M \) is assumed noncompact throughout.