Singularities in the Kerr–Schild Metrics

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Abstract

It is shown that the only empty space solution of the type "flat space plus the square of a null vector" whose singularities are confined to a bounded region is the Kerr metric.

The Kerr–Schild metrics have the form

\[ g_{ab} = \eta_{ab} + 2h_l^a l^b \]  \hspace{1cm} (1)

where \( g^{ab} l_a l_b = 0 \) and \( (\eta_{ab}) = \text{diag} (1, -1, -1, -1) \) in Cartesian \((t, x, y, z)\) coordinates. It follows that

\[ \det (g_{ab}) = \det (\eta_{ab}) = -1 \]
\[ g^{ab} = \eta^{ab} - 2h_l^a l^b \]
\[ l^a = g^{ab} l_b = \eta^{ab} l_b \]

and so \( l \) is also null with respect to the auxiliary flat metric:

\[ \eta^{ab} l_a l_b = 0 \]

We shall use spinor coordinates

\[
\begin{pmatrix}
    x^{00} & x^{01} \\
    x^{i0} & x^{i1}
\end{pmatrix} =
\begin{pmatrix}
    u & \xi \\
    \bar{\xi} & v
\end{pmatrix} = 2^{-1/2}
\begin{pmatrix}
    t - z & x + iy \\
    x - iy & t + z
\end{pmatrix}
\]
giving the flat part of the metric as
\[ ds_0^2 = 2 \, du \, dv - 2 \, d\xi \, d\bar{\xi} \]
The real null vector \( l \) (taken as future pointing) determines a spinor, \( \kappa \), by
\[ l^\alpha \beta = \kappa^\alpha \kappa^\beta \]
(the dot indicating complex conjugate and the indices ranging over 0 and 1). The multiplicative arbitrariness may be removed by writing
\[ (\kappa^0, \kappa^1) = (1, Y) \]
where \( Y \) is an arbitrary complex function of position. This spinor and the linearly independent
\[ (i^0, i^1) = (0, 1) \]
generate a flat space null spin tetrad:
\[ l = \kappa^\alpha \kappa^\beta \partial_{\alpha \beta} = \partial_u + Y \partial_\xi + \overline{Y} \partial_{\bar{\xi}} + Y \overline{Y} \partial_v \]
\[ n_0 = i^\alpha i^\beta \partial_{\alpha \beta} = \partial_v \]
\[ m = \kappa^\alpha i^\beta \partial_{\alpha \beta} = \partial_\xi + \overline{Y} \partial_v \]
\[ \overline{m} = i^\alpha \kappa^\beta \partial_{\alpha \beta} = \partial_{\bar{\xi}} + Y \partial_v \]
of which \( l \) and \( n_0 \) are real, and \( m \) and \( \overline{m} \) complex conjugates.
The covariant spinor components are taken as
\[ (\kappa_0, \kappa_1) = (\kappa^1, -\kappa^0) = (Y, -1) \]
\[ (i_0, i_1) = (1, 0) \]
so that the corresponding flat null differential forms are
\[ \lambda = \kappa_\alpha \kappa_\beta dx^{\alpha \beta} = du - \overline{Y} d\xi - Y d\bar{\xi} + Y \overline{Y} du \]
\[ \nu_0 = i_\alpha i_\beta dx^{\alpha \beta} = du \]
\[ \mu = \kappa_\alpha i_\beta dx^{\alpha \beta} = -d\bar{\xi} + \overline{Y} du \]
\[ \overline{\mu} = i_\alpha \kappa_\beta dx^{\alpha \beta} = -d\xi + Y du \]
The metrics are
\[ ds_0^2 = 2\lambda \nu_0 - 2\mu \overline{\mu} \]
and
\[ ds^2 = 2\lambda (\nu_0 + h\lambda) - 2\mu \overline{\mu} \]
A null tetrad for the latter is constructable merely by changing \( \nu_0 \) to
\[ \nu = du + h\lambda \]
and hence \( n_0 \) to
\[ n = \partial_v - h \xi \]