Vacuum Type N Space-times Admitting Homothetic Vector Fields with Isolated Fixed Points

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We consider vacuum space-times $(M, g)$ which are of Petrov type N on an open dense subset of $M$, and which admit (proper) homothetic vector fields with isolated fixed points. We prove that if such is the case then, at the fixed point, $(M, g)$ is flat and the homothetic bivector, $X_{[e, f]}$, is necessarily simple-timelike. Furthermore, we prove that if the homothetic bivector remains simple-timelike in some neighbourhood of the fixed point then, around the fixed point, the space-time in question is a pp-wave. The paper ends with a local characterization and some examples of space-times satisfying these conditions.

1. INTRODUCTION

Let $(M, g)$ be a space-time, i.e. a smooth, 4-dimensional paracompact manifold $M$ together with a metric $g$ of Lorentzian signature $(-, +, +, +)$. A vector field $X$ on $M$ is said to be a (proper) homothetic vector field if there exists a real number $\lambda \neq 0$ such that, $\mathcal{L}$ denoting the Lie derivative, we have

$$\mathcal{L}Xg = 2\lambda g. \quad (1)$$

$\lambda$ is then called the homothetic constant (of $X$). By means of a rescaling, one can always set $\lambda = 1$, and this will be assumed done from now on. Setting

$$f_{ab} = \frac{1}{2}(X_{a,b} - X_{b,a}), \quad (2)$$

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where the semi-colon denotes the covariant derivative with respect to the metric connection, one has \( X_{a;b} = g_{ab} + f_{ab} \). The bivector \( f_{ab} \) is called the homothetic bivector (of \( X \)).

A point \( p_0 \in M \) is said to be a fixed point of \( X \) if one has \( X_{p_0} = 0 \). As is well known (\( X \) being an affine vector field, see Ref. 1) if one considers at \( p_0 \) a real null tetrad which spans the blades of \( f \) and one considers the normal coordinate system associated with this tetrad, then, with respect to these coordinates \( X \) admits the local expression \( X_p = M \times q \), where \( M \) is the matrix \( \delta^a_b + f^a_b \) computed at \( p_0 \) and \( p \) is identified with the tangent vector \( \exp^{-1}(p) \). This shows that the classification of the homothetic bivector \( f_{ab} \) at \( p_0 \) is in fact a classification of \( X \) around \( p_0 \) and provides us with some essential information about the local structure of \((M, g)\) around \( p_0 \). Following this point of view, and using a result of Beem [5], one proves then that if \( f \) is 0, simple-null or simple-spacelike at \( p_0 \) then \((M, g)\) is flat around \( p_0 \) [2,6]. This leaves the cases when \( f \) is either simple-timelike or non-simple to be analysed. In both cases, one can choose at \( p_0 \) a real null tetrad \((l, m, x, y)\) such that one has, at \( p_0 \)

\[
f_{ab} = 2\alpha_0 l_{[a}m_{b]} + 2\beta_0 x_{[a}y_{b]},
\]

(3)

where \( \alpha_0, \beta_0 \in \mathbb{R} \) and one has \( \alpha_0 \neq 0 \) in all cases and \( \beta_0 \neq 0 \) in the non-simple case. The value of \( \alpha_0 \) plays an essential role; in fact if \( |\alpha_0| < \lambda = 1 \), one can again use Beem’s result to prove that \((M, g)\) is flat around \( p_0 \) [2]. If \( |\alpha_0| = \lambda = 1 \), the matrix \( M \) described above, has rank 3 at the fixed point, and it follows that the (sufficiently small) elements in its kernel correspond (through the exponential map) to fixed points of \( X \); in fact this kernel is a null subspace of \( T_{p_0}M \) and so (at least around \( p_0 \)) the set of fixed points of \( X \) is (part of) a null geodesic. This case has been analysed by Alekseevski [6] and Hall [2], who have proved that, around \( p_0 \), \((M, g)\) is a plane-wave.

When \( |\alpha_0| > 1 \), \( M \) has rank 4 at \( p_0 \) and so \( p_0 \) is an isolated fixed point of \( X \). No general results concerning the local structure of \((M, g)\) around \( p_0 \) are known in this case. However, it should be stressed that, at \( p_0 \) and on some subsets of \( M \) containing \( p_0 \), a complete classification, in what concerns Petrov and Segre types has been obtained by Hall [2] and we summarize them now.

Consider the normal coordinates \((u, v, r, s)\) associated with the tetrad \((l, m, x, y)\) at \( p_0 \). As follows from the expression of the matrix \( M \) described above, the local expression for \( X \) in these coordinates is then

\[
X = (1 - \alpha_0)u \partial_u + (1 + \alpha_0)v \partial_v + (r + \beta_0 s) \partial_r + (s - \beta_0 r) \partial_s.
\]

(4)

Suppose that \( \alpha_0 > 1 \) (if \( \alpha_0 < -1 \), change \( u \) with \( v \) in what follows) and denote by \( S \) the hypersurface \( u = 0 \), by \( \Gamma \) the curve given by \( v = r = s = 0 \).