Inclusion of the four-derivative terms $f R_{\mu \nu} R^{\mu \nu} (-g)^{1/2}$ and $f R^2 (-g)^{1/2}$ into the gravitational action gives a class of effectively multimass models of gravity. In addition to the usual massless excitations of the field, there are now, for general amounts of the two new terms, massive spin-two and massive scalar excitations, with a total of eight degrees of freedom. The massive spin-two part of the field has negative energy. Specific ratios of the two new terms give models with either the massive tensor or the massive scalar missing, with correspondingly fewer degrees of freedom. The static, linearized solutions of the field equations are combinations of Newtonian and Yukawa potentials. Owing to the Yukawa form of the corrections, observational evidence sets only very weak restrictions on the new masses. The acceptable static metric solutions in the full nonlinear theory are regular at the origin. The dynamical content of the linearized field is analyzed by reducing the fourth-order field equations to separated second-order equations, related by coupling to external sources in a fixed ratio. This analysis is carried out into the various helicity components using the transverse-traceless decomposition of the metric.

§1: Introduction

Upon several occasions, dating back to the early days of general relativity, it has been suggested that the Einstein equations for the gravitational field be replaced by others involving derivatives of higher than second order. Early suggestions [1] by Weyl and Eddington concerned an attempt to include the electromagnetic field in a unified geometrical framework, but this line of approach proved unfruitful and was eventually abandoned. Higher-derivative theories were studied in the general context of quantum field theory by Pais and
Uhlenbeck in 1950 [3]. Later on, quantum field theory also gave rise to suggestions that higher-derivative terms be included in the gravitational Lagrangian in order to permit renormalization of divergences in the quantum corrections to the interactions of matter fields [4]. This motivation was further reinforced in recent years by the demonstration that gravitation itself contains nonrenormalizable divergences when in interaction with matter, [6].

The inclusion of terms proportional to $R_{\mu\nu}R^{\mu\nu}$ and $R^2$ in the gravitational Langrangian produces a stabilization of the divergence structure of gravity, allowing it to be renormalized, along with its matter couplings. The details of this renormalization have been discussed elsewhere, [7] and will not be considered here. In the present article, the classical features of such higher-derivative models will be discussed.

The early investigators emphasized that the empty space solutions of Einstein's equations, $R_{\mu\nu} = 0$, are also solutions of the field equations derived from actions like $f(-g)^{1/2}R_{\mu\nu}R^{\mu\nu}$ or $f(-g)^{1/2}R^2$, and thus they thought that all the classical tests of general relativity were automatically satisfied. It was not until comparatively recently [8] that it was pointed out that although, e.g., the Schwarzschild solution is a solution to the empty space equations, it is not the one that couples to a positive definite matter distribution. In fact, those solutions of purely four-derivative models which do couple to a positive matter source are not asymptotically flat at infinity. One may see this immediately from the linearized theory, where the Green’s function is generically $(\nabla^2 \nabla^2)^{-1} \sim r$. Similarly, the Schwarzschild solution is ruled out because $\nabla^2 \nabla^2 (r^{-1}) \sim \nabla^2 \delta^3 (x)$, which does not correspond to a positive definite matter distribution.

Accordingly, in this work, we shall restrict ourselves to models derived from actions that include both the Hilbert action $f(-g)^{1/2}R$ and the four-derivative terms. There are only two independent additions that one can possibly make, because of the Gauss–Bonnet relation in four dimensions, $(-g)^{1/2}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) = \text{div}$, so that we have only a two-parameter family of field equations (we do not consider a cosmological term). We write the action in the form

$$I = -\int (-g)^{1/2}(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \gamma \kappa^{-2}R)$$

(1.1)

where $\kappa = 32\pi G$, and $\alpha$, $\beta$, and $\gamma$ are dimensionless numbers. It will turn out that the correct physical value for $\gamma$ is 2, as in Einstein’s theory.

We shall find that the static spherically symmetric solutions to the field equations derived from (1.1) either reduce asymptotically to the sum of a Newtonian and two Yukawa potentials, or they are unbounded at infinity and must be eliminated by boundary conditions. The masses in these Yukawa potentials are only weakly constrained by the observational evidence to be $\geqslant 10^{-4}$ cm$^{-1}$. Although the magnitude of these effects is negligible at laboratory or astronomical distances, there are some interesting qualitative features. Coupling the lin-

4More recent use of higher derivatives to regularize the stress tensor is reviewed in [5].