Stability of Certain Spatially Homogeneous Cosmological Models

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Abstract

We investigate the general behavior of spatially homogeneous cosmological models at large times. Using existing techniques from stability theory we discover that some known exact solutions are asymptotically stable despite possessing special properties. Some consequences of these results are then discussed.

§(1): Introduction

One of the most challenging problems of classical general relativity has been the issue of the general solution to Einstein's equations in the neighborhood of a space-time singularity [1]. Detailed studies of spatially homogeneous cosmological models have revealed the presence of chaotic dynamical behavior [2] near the initial singularity which prevents the deployment of traditional series approximation techniques as a route to elucidating the general inhomogeneous behavior. Here, we would like to investigate a complementary, but more tractable problem: what is the behavior of the general cosmological solutions to Einstein's equations at late cosmic times, far from any singularity?

Unlike the problem of behavior near the initial singularity, an answer to this question has immediate observational consequences for the present structure of the microwave background temperature anisotropy [3]; moreover, it is compromised neither by uncertainties regarding the truth of general relativity at energies exceeding 10^{19} GeV nor by the presence of quantum matter fields, and is obtainable by existing methods of investigating differential equations.

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As in studies of the general solution close to space-time singularities, the obvious starting point for any investigation of late-time asymptotic behavior is the spatially homogeneous cosmological models [4]. We shall show that these supply a number of paradigms from which a rigorous analysis of the general inhomogeneous solution can sensibly begin. Of the most general class of homogeneous cosmologies, which require four free constants to completely specify the metric in vacuum, only three special exact solutions are known. They are all vacuum solutions and are of Bianchi types $VI_{-1/9}$, $VI_h$, and $VII_h$. The metrics of these solutions are as follows.

(A) The Collinson-French $VI_{-1/9}$ Solution. This vacuum solution [5] contains no arbitrary constants in the metric:

$$ds^2 = dt^2 - \frac{75}{8} t^2 dz^2 - 5 t^{6/5} e^{-z} dx dz - \frac{6}{5} t^{2/5} e^{-2z} dx^2 - t^{6/5} e^{4z} dy^2$$

(1)

It has other interesting features; in particular, it possesses a nonscalar initial singularity with all curvature invariants finite there.

(B) The Ellis-MacCallum $VI_h$ Solution. This vacuum solution [6] contains two arbitrary constant parameters $(k, c)$ only and is a space-time with metric $k^2 ds^2 = \left(\frac{\sinh 2t}{\cosh t}\right)^{1/2} (dt^2 - dz^2) - e^{2z}$

where

$$F = \left(\frac{\sinh 2t}{\cosh t}\right)^{1/2}$$

and the label $h$ is given by $h = -c^2$.

(C) The Lukash $VII_h$ Solution. This vacuum solution [7] contains two arbitrary constant parameters $(\lambda, k)$ only and is a plane wave with metric

$$ds^2 = dt^2 - c^2(t) dz^2 - g_{\alpha\beta} dx^\alpha dx^\beta$$

(4)

where $1 \leq \alpha, \beta \leq 2$ and $g_{\alpha\beta}$ is the $2 \times 2$ matrix

$$g_{\alpha\beta} = \begin{bmatrix} a^2 e^{2z} \left[\cosh \mu + \sinh \mu \cos (kz + \phi)\right], & a^2 e^{2z} \sinh \mu \sin (kz + \phi) \\ a^2 e^{2z} \left[\sinh \mu \sin (kz + \phi)\right], & a^2 e^{2z} \left[\cosh \mu - \sinh \mu \cos (kz + \phi)\right] \end{bmatrix}$$

(5)

As stressed by Siklos [4] the $VI_{-1/9}$ solutions are as general as the $VI_h$ and $VII_h$ solutions in vacuum because of the degeneracy of constraint equations when $h = -1/9$. 

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