A Characterization of Robertson-Walker Spaces by Null Sectional Curvature

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Abstract

For $N$ a null vector and $A$ a vector perpendicular to $N$, define the null sectional curvature, with respect to $N$, of the plane $N \wedge A$ as $\kappa_N(N \wedge A) = \langle R(N, A)A, N \rangle / \langle A, A \rangle$. Then Robertson-Walker metrics can be locally characterized as those for which $\kappa_N$ at each point is a constant for all the null plans at that point (in each null direction, $N$ must be appropriately chosen). A global characterization of Robertson-Walker spaces is achieved by adding completeness and causality hypotheses.

Sectional curvature, although a powerful tool in Riemannian geometry, plays a somewhat different role in Lorentz manifolds (signature $-+\cdots+$), in part because sectional curvature is not defineable for null planes ($2$-planes degenerate in the induced metric). An additional tool for the study of such planes is null sectional curvature, defined in [1] as follows: Let $\pi$ be any null plane; $\pi$ consists of a one-dimensional subspace of null vectors and of spacelike vectors perpendicular to that subspace. Let $N$ be one of the null vectors and $A$ one of the spacelike vectors. Then the null sectional curvature of $\pi$ with respect to $N$ is

$$\kappa_N(\pi) = \frac{\langle R(A, N)N, A \rangle}{\langle A, A \rangle}$$

this is independent of the choice of a spacelike vector $A$ in $\pi$. Here $\langle, \rangle$ is the metric and $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. The metric on the bivectors $\Lambda^2M$ of $M$ will also be denoted with $\langle, \rangle$: $\langle A \wedge B, C \wedge D \rangle = \langle A, C \rangle \langle B, D \rangle$. 

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\( \langle A, D \rangle \langle B, C \rangle \); note that a 2-plane \( \pi \), considered as a bivector, is null if and only if \( \langle \pi, \pi \rangle = 0 \).

The purpose of this paper is to investigate the question, what happens if, in an appropriate sense, null sectional curvature is a point function, i.e., at each point it is independent of the choice of null planes at that point? Under the right conditions the answer is that the manifold is a Robertson-Walker (also known as Friedmann) space, i.e., a warped product \( (\mathbb{R}^1, -dt^2) \times _\rho (V, h) \) with \( (V, h) \) a Riemannian manifold of constant curvature (see [2], pp. 134-142); the notation used here is that for \( (M_1, g_1) \) and \( (M_2, g_2) \) pseudo-Riemannian manifolds and \( \rho: M_1 \to \mathbb{R}^+ \) a positive scalar function, \( (M_1, g_1) \times _\rho (M_2, g_2) = (M_1 \times M_2, p_1^*g_1 + (\rho \circ p_1)p_2^*g_2) \), where \( p_i: M_1 \times M_2 \to M_i \) is projection.

Since null sectional curvature depends on the choice of one element of a one-dimensional space of null vectors, it is first necessary to fix this choice. If all null planes containing a given null direction are to be measured by the same null vector, then one must make one choice for each null direction: For a Lorentz manifold \( M^n \) \((n \geq 3)\), define a null congruence on \( M \) to be a submanifold \( C \) of the bundle \( T^0M \) of nonzero null vectors on \( M \) such that for all \( N \in T^0M \), there is exactly one scalar \( \lambda \) with \( \lambda N \in C \); for \( p \in M \), let \( C_p = C \cap T_pM \).

The question raised above can now be put a little more precisely: What happens if, for an appropriate congruence \( C \), there is a scalar function \( k: M \to \mathbb{R} \) such that for any null plane \( \pi \) at \( p \in M \), containing \( N \in C_p \), \( k_N(\pi) = k(p) \)?

What makes a null congruence appropriate? One convenient property would be that the congruence respects null geodesics: Define a null congruence to be geodesic if for all null geodesics \( \beta(\nabla_\beta \dot{\beta} = 0) \), if \( \dot{\beta}(t_0) \in C \) for some \( t_0 \), then \( \dot{\beta}(t) \in C \) for all \( t \).

How may a null congruence be easily specified? One way is with a timelike vector field: For \( U \) a timelike vector field on a Lorentz manifold \( M \), define the null congruence \( C(U) \) associated with \( U \) by

\[
C(U) = \{ N \in T^0M \mid \langle N, U \rangle = 1 \}
\]

Since a timelike and a null vector always have a nonzero inner product, this will always be a null congruence (timelike is understood to preclude being zero).

Let us first examine the situation of null sectional curvature being a point function with respect to \( C(U) \) for \( U \) any timelike field; we may just as well consider the congruence \( C(E) \) for \( E = U/|U| \) \(|X| = |\langle X, X \rangle|^{1/2} \), since that changes each choice of null vector \( N \) and, hence, each \( k_N(\pi) \) by a scalar function. It turns out (Proposition 1) that this is equivalent to infinitesimal isotropy of \( M \) with respect to \( E \), as defined by Karcher in [3]: \( M \) is infinitesimally isotropic with respect to a unit timelike vector field \( E \) is the Riemann curvature tensor \( R \), regarded as a skew-adjoint endomorphism of the bivectors \( \Lambda^2 M \) of \( M \), has two eigenspaces: the bivectors perpendicular to \( E \), denoted \( \Lambda^2(E \perp) \), and the 2-planes containing \( E \), denoted \( \Lambda^2E \). It will be convenient to recall here one of Karcher's results in [3]: