AN ACTION PRINCIPLE FOR THE MOTION OF PARTICLES

P.A.M. DIRAC

Physics Department, Florida State University,
Tallahassee, Florida

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ABSTRACT

An action principle is set up for the motion of charged matter in the presence of the Einstein gravitational field. It can be applied to the motion of an extended particle and avoids the difficulties of a point particle arising from the singularities in the field.

§(1): MOTION OF A POINT PARTICLE

For the Einstein gravitational field interacting with the electromagnetic field the action $I_F$ is given by

$$16\pi I_F = \int (R - F_{\mu\nu}E^{\mu\nu})\sqrt{g} \, d^4x,$$

where $\sqrt{g}$ is short for $\sqrt{-g}$ and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. The action is a function of the $g_{\mu\nu}$ and their first and second derivatives and the first derivatives of the $A_\mu$. If we vary the $g_{\mu\nu}$ and the $A_\mu$, we get by standard methods

$$16\pi \delta I_F = \int \left\{ - (R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi E^{\mu\nu})\delta g_{\mu\nu} + 4F^{\mu\nu}_{;\nu}\delta A_\mu \right\}\sqrt{g} \, d^4x,$$

where $E^{\mu\nu}$ is the electromagnetic energy tensor,

$$4\pi E^{\mu\nu} = - F^{\mu\nu}_{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}g^{\mu\nu}R. \tag{1.2}$$

By putting $\delta I = 0$, we get

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi E^{\mu\nu} = 0, \tag{1.3}$$

$$E^{\mu\nu}_{;\nu} = 0. \tag{1.4}$$
which are the field equations in the absence of matter and charge.

Now introduce a point particle with the coordinates $z^\mu$, functions of the proper time $s$ measured along its world-line. Its velocity vector is $v^\mu = dz^\mu/ds$. Let $m$ be its mass and $e$ its charge. The action for the particle is an integral along the world-line,

$$I_P = - \int (m + eA_\mu v^\mu) ds. \quad (1.5)$$

If we vary the world-line, displacing each point of it by $\delta z^\sigma$, we get, again by standard methods,

$$\delta I_P = \left\{ m \frac{d z^\mu}{ds} + m \Gamma^\mu_{\alpha \beta} v^\alpha v^\beta + e F^\mu_{\nu} v^\nu \right\} g_{\mu \sigma} \delta z^\sigma ds.$$

Putting this equal to zero, we get the equation of motion for the particle,

$$m \frac{d z^\mu}{ds} + m \Gamma^\mu_{\alpha \beta} v^\alpha v^\beta + e F^\mu_{\nu} v^\nu = 0. \quad (1.6)$$

The trajectory is a geodesic, modified by the Lorentz force.

The above calculation is valid when $m$ and $e$ are small, so that their influence on the field is negligible. The particle is then of the nature of a test particle.

If $m$ and $e$ are not small, the influence of the particle on the field must be taken into account. The equations for the field and the particle all follow from the comprehensive action principle $\delta (I_F + I_P) = 0$. The variations of $g_{\mu \nu}$ and $A_\mu$ in $I_P$ give rise to further terms that must be included in equations (1.3,4). These further terms vanish except on the world-line and cause singularities in the field at the world-line.

For the variation principle to be valid under these circumstances, one must arrange things so that the displacement of the particle is necessarily accompanied by the displacement of the singularity in the field. This requires that one must choose suitable working variables $q$, such that the variation of the action is a linear function of the $\delta q$'s. An example of this method (for the Born-Infeld electrodynamics) has been given by the author \[1\].

With the standard Einstein theory, one can get a simpler and more direct method by departing from the point particle, replacing it by a particle of finite size. It is possible to get simple equations which do not depend on the structure of the extended particle.