SU_q(2) Covariant \( \hat{R} \)-Matrices for Reducible Representations

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Abstract. We consider SU_q(2) covariant \( \hat{R} \)-matrices for the reducible 3 \( \oplus \) 1 representation. There are three solutions to the Yang–Baxter equation. They coincide with the previously known \( \hat{R} \)-matrices for SO_q(3) and SO_q(3, 1). Also, they are the three \( \hat{R} \)-matrices which can be constructed by using four different SU_q(2) doublets. Only two of the three \( \hat{R} \)-matrices allow a differential structure on the reducible four-dimensional quantum space.


1. Introduction

From the universal \( R \)-matrix of the SU_q(2) algebra, there follows an \( \hat{R} \)-matrix for the product of any two irreducible representations of SU_q(2). However, these \( \hat{R} \)-matrices do not allow a differential structure on the representation spaces with angular momentum two or higher. The reason is that in the decomposition of the \( \hat{R} \)-matrix into projectors on irreducible subspaces of the product space, all projectors enter with different eigenvalues. It is then not possible to build a differential structure based on this \( \hat{R} \)-matrix that would satisfy the Poincaré–Birkhoff–Witt theorem. The situation is exactly the same if one tries to define creation and annihilation operators for particles with angular momentum two or larger. Thus, it is not possible to construct a Fock space based on the universal \( \hat{R} \)-matrix.

There are, however, \( \hat{R} \)-matrices defined on reducible representations of SU_q(2) that are SU_q(2) covariant but do not decompose the same way as the SU_q(2) generators do. If in the reduction of the product the same irreducible representation occurs several times, the generators and the \( \hat{R} \)-matrix mix them differently. Among these \( \hat{R} \)-matrices, there are candidates for a differential structure. The problem is that these reducible \( \hat{R} \)-matrices are not known in general.

In this Letter, we give an example of such \( \hat{R} \)-matrices and construct them by a method that could be generalized to higher spin. The emphasis of this Letter, however, is a search for all possible \( \hat{R} \)-matrices for a given reducible representation – in our case a representation with one triplet and one singlet under SU_q(2).
This Letter is organized as follows. In the next section, we discuss the notion of SUq(2) covariance and R-matrices. After two simple examples of irreducible representations we solve the Yang–Baxter equation for the 3 ⊕ 1 representation and find exactly three classes of solutions. In the third section, comparison is made with the q-spinor approach and we find that the three classes of R-matrices can be constructed from the q-spinors as well. In the final section, we discuss the suitability of these R-matrices for a differential calculus.

2. ~U_q(SU_q(2)) and Covariance

We begin by describing the algebra ~U_q(SU_q(2)). It is generated by the generators T+, T- and T3 which obey the relations

\[ q^{-1}T^+T^- - qT^-T^+ = T^3, \]
\[ q^2T^3T^+ - q^{-2}T^+T^3 = (q + q^{-1})T^+, \]
\[ q^{-2}T^3T^- - q^2T^-T^3 = -(q + q^{-1})T^-, \]

(2.1)

where q is the deformation parameter. ~U_q(SU_q(2)) is a Hopf algebra and as such has a coproduct Δ. It essentially describes how the generators act on a tensor product of representation spaces. For ~U_q(SU_q(2)), we have

\[ Δ(T^+) = T^+ ⊗ 1 + (1 - λT^3)^{1/2} ⊗ T^+, \]
\[ Δ(T^-) = T^- ⊗ 1 + (1 - λT^3)^{1/2} ⊗ T^-, \]
\[ Δ(T^3) = T^3 ⊗ 1 + (1 - λT^3) ⊗ T^3. \]

Here we define λ = q - q^{-1}. This coproduct is a homomorphism of the algebra (2.1) and is coassociative.

The representations of ~U_q(SU_q(2)) are well known [1]. As in the classical case there is a Casimir operator with eigenvalues labeled by j, the total angular momentum. States within each representation have eigenvalues of T3 labeled by m. The action of the generators on a state [j, m] is

\[ T^+ [j, m] = q^{-1} \sqrt{[j + m + 1]_{q^{-2}} [j - m]_{q^2}} [j, m + 1], \]
\[ T^- [j, m] = q \sqrt{[j + m]_{q^{-2}} [j - m + 1]_{q^2}} [j, m - 1], \]
\[ T^3 [j, m] = q^{-1} [2m]_{q^{-2}} [j, m]. \]

[\[n\]]_r is the q-number defined by [\[n\]]_r = (r^n - 1)/(r - 1).

We will be interested in tensor products of these representation spaces. Consider two different sets of states [j_1, m_1] and [j_2, m_2]. As in the classical case, their tensor product can be written as a sum of states with different total angular momentum:

\[ [j_1, m_1] ⊗ [j_2, m_2] = \sum_{J = |j_1 - j_2|}^{j_1 + j_2} C_q(j_1, m_1, j_2, m_2, J) [J, m_1 + m_2]_{12}. \]

(2.4)