One-Sided Type-D Gravitational Instantons

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Abstract

It is shown that for every one-sided type-D gravitational instanton, Einstein's vacuum equations can be reduced locally to a single second-order, nonlinear differential equation of second degree of one real function.

§1: Introduction

It is well known that for every complex space-time with one-sided algebraically degenerate Weyl tensor, Einstein's vacuum equations can be reduced to a single nonlinear differential equation of the second order and second degree. (1) In particular, for $H$ spaces this equation appears to be the "second heavenly equation." (2, 3)

On the other hand, every self-dual (or anti-self-dual) vacuum gravitational instanton is locally a Kählerian manifold and in this case Einstein's equations are reducible locally to a single second-order, nonlinear differential equation of second degree (4, 5) which is the analog of Plebański's "first heavenly equation." (2, 3) Hence, the natural question arises ‘can the result of Ref. 1 concerning complex space-times be carried over to the spaces of positive-definite metric'? Our paper is devoted to this subject.

Using the "instanton" version of the generalized Goldberg-Sachs Theorem (5, 6) we prove that Einstein's vacuum equations for the one-sided type-D
gravitational instanton, which appears now to be a locally Hermitian manifold, can be reduced locally to a single second-order, nonlinear differential equation of second degree of one real function. This function is closely related to the "first key function" introduced by Plebański in his studies over \( H \) Spaces. (2, 3)

Our considerations are purely local. The formalism used is the null tetrad one as presented in Refs. 4–7.

\section{(2): The Reduction of Einstein Equations}

We assume that the gravitational instanton \( M \) is of the type anything \( \times D \) and moreover, that its traceless Ricci tensor \( C_{\alpha\beta} \) vanishes. Then from the instanton version of the generalized Goldberg-Sachs theorem (5, 6) it follows that the metric \( ds^2 \) of \( M \) is locally Hermitian, i.e., for some local complex coordinates \( \{z^\alpha\}, \alpha = 1, 2, \)

\[
\begin{aligned}
ds^2 &= g_{\alpha\bar{\alpha}} \, dz^\alpha \otimes \overline{dz^\alpha} + g_{\bar{\alpha}\alpha} \, \overline{dz^\alpha} \otimes dz^\alpha \\
\overline{\bar{\beta}} &= \overline{1}, \overline{2}, \quad \overline{dz^\beta} = \overline{dz^\bar{\beta}}, \quad g_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta}
\end{aligned}
\]

Define four 1-forms:

\[
\begin{aligned}
e^1 &= g_{1\bar{\beta}} \, dz^\bar{\beta}, \quad e^4 = g_{2\bar{\beta}} \, dz^\bar{\beta} \\
e^2 &= dz^1, \quad e^3 = dz^2
\end{aligned}
\]

Then

\[
ds^2 = e^1 \otimes e^2 + e^2 \otimes e^1 + e^3 \otimes e^4 + e^4 \otimes e^3
\]

and we find that the four 1-forms \( \{e^a\}, a = 1, 2, 3, 4, \) constitute an extended null tetrad (see Ref. 5). Now, from the first Cartan structure equations

\[
de^a + \Gamma^a_{\,b} \wedge e^b = 0
\]

one can obtain the "dotted" connection forms:

\[
\begin{aligned}
\Gamma_{41} &= \frac{1}{g} \, g_{\alpha[2,\bar{1}]} \, dz^\alpha \\
\Gamma_{32} &= g_{\alpha[2,1]} \, dz^\alpha \\
-\Gamma_{12} + \Gamma_{34} &= -g^{\alpha\bar{\beta}} \, g_{\alpha[\bar{\beta},\gamma]} \, dz^\gamma - g^{\bar{\beta}\alpha} \, g_{\bar{\beta}(\bar{\alpha},\gamma)} \, dz^\gamma
\end{aligned}
\]

where \( g := \det (g_{\alpha\bar{\beta}}) \), \( g^{\alpha\bar{\beta}} \) is a prime (') denotes the partial derivative. For the base defined by (2) and with \( C_{\alpha\beta} = 0 \), the "dotted" \( \equiv \) anti-