A CRITERION FOR THE STABILITY OF A CHARGED SPHERE IN GENERAL RELATIVITY

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ABSTRACT

A criterion for stability of a charged sphere is obtained by use of two different methods. The result is applied to a charged dust in order to investigate its stability.

§(1): INTRODUCTION

The Reisner-Nordström solution [1] is the unique, asymptotically flat and spherically symmetric exterior solution of the Einstein-Maxwell equations. The corresponding interior solution for a spherically symmetric charged sphere has been studied by many people in connection with a model of the electron [1,2,3,4], of a charged star, etc., in which the problem of stability of a sphere [5,6] is very important. The stability of a non-charged sphere has been studied in detail by Chandrasekhar [7] using the dynamical method and by Wheeler et al. [8] using the variational method.

In this paper we shall derive a criterion for the stability of a charged sphere by the following two methods: a given system is regarded as stable (a) when the total energy (including the energy of the gravitational field of the system) has its minimum value under any variation of the inner configuration (the variational method), and (b) when an adiabatic, radial, small oscillation of the system will not develop any further (the dynamical method).

The conditions for stability obtained in sections 2, 3 in these two different ways are shown to be identical. In section 4 the result will be applied to discuss the problem of the stability of a charged dust.

§(2): VARIATION OF THE TOTAL ENERGY

In this section we shall obtain the criterion for the stability of a charged sphere using the variational method. Various definitions of the energy density (including that of the gravitational field) have been given by many authors [9-12].
In the following we adopt the Komar's definition [12], which may be the most satisfactory, since its integral over the total 3-space is independent of the choice of space coordinates. He gives the conserved vector density defined by

\[ \mathfrak{P}^i(\xi^j) = \frac{\sqrt{-g}}{8\pi} g^i_p g^{ln} [\xi^m (g_{mn,p} - \xi_{mp,n}) + (\xi^m_p g_{nn} - \xi_{mn}^m g_{mp})] \]  

(2.1)

where \( \xi^i \) is an arbitrary vector field which indicates an infinitesimal coordinate transformation. We can rewrite \( \mathfrak{P}^i(\xi) \) as

\[ \mathfrak{P}^i(\xi^j) = \frac{\sqrt{-g}}{8\pi} (\xi^i_{;k} - \xi^k_{;i}) \xi^j, \]  

(2.2)

which satisfies the conservation relation

\[ \partial_k \mathfrak{P}^j(\xi^j) = 0. \]  

(2.3)

The total energy \( P(\delta_0^k) \) of a given system is expressed in terms of \( \mathfrak{P}^j(\delta_0^k) \) as follows;

\[ P(\delta_0^k) = \int_{E=\text{constant}} \mathfrak{P}^j(\delta_0^k) d^3x. \]  

(2.4)

In the case of a charged sphere we can substitute in (2.4) the Reisner-Nordström metric [7] given by

\[ ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)} + r^2 d\Omega^2, \]  

(2.5)

where \( M \) and \( Q \) are, respectively, the mass and the charge of the sphere. In this way we obtain the result

\[ P(\delta_0^k) = M. \]  

(2.6)

The mass \( M \) is, of course, a functional of the inner-energy distribution \( \varepsilon \) and the charge distribution \( \rho \), and its explicit expression can be obtained when we know the inner solution of the Einstein-Maxwell equations for this case.

A spherically symmetric and static inner metric [10] can be written, in general, in the form:

\[ ds^2 = - e^\upsilon dt^2 + e^\lambda dr^2 + r^2 d\Omega^2, \]  

(2.7)