GENERAL SOLUTION OF A 2-D WEAK SINGULAR INTEGRAL EQUATION WITH CONSTRAINT AND ITS APPLICATIONS

Yun Tianquan (云天铨)

(Received Jan. 22, 1996; Revised Feb. 10, 1996)

Abstract

In this paper, the solution, more general than [1], of a weak singular integral equation

\[ \int_{\Omega} \int_{0}^{\infty} p(s, \psi) ds \psi d\psi = F(r, \theta), \quad (r, \theta) \in Q = Q + \partial Q \]

subject to constraint

\[ p(s, \psi) = 0, \text{ for } (s, \psi) = (r, \theta) \in Q = \{(r, \theta) | F(r, \theta) > c_0\} \]

is found

\[ p = \frac{2}{\pi} \left[ \sqrt{w} g'(0) + \int_{0}^{\pi} \sqrt{w - u} g'(u) du \right] \]

where \( h \) and \( F \) are given continuous functions; \( (s, \psi) \) is a local polar coordinating with origin at \( M(r, \theta) \); \( (r, \theta) \) is the global polar coordinating with origin at \( O(0, 0) \); \( F(\psi, \theta) = c_0 \) (const.) is the boundary contour \( \partial Q \) of the considered range \( Q, g(w) = F(r, \theta) / [\pi h(\psi_0)], g' = dg/d\omega, w = N - r^2 \sin^2(\theta + \psi_0), \psi_0 \) and \( N \) are mean values. The solution shown in type (2.19) of [1] is a special case of the above solution and only suits \( F(r, \theta) = w \). The solution of a rigid cone contact with elastic half space, more simple and clear than Love's (1939), is given as an example of application.

Key words Radon transform, Abel integral equation, theorem mean value, Hertz's solution

I. Introduction

The integral equation

\[ \int_{\Omega} \int_{-\infty}^{\infty} p(s, \psi) ds \psi d\psi = F(r, \theta), \quad (r, \theta) \in Q = Q + \partial Q \quad (1.1) \]

subject to constraint

\[ p(s, \psi) = 0, \text{ for } (s, \psi) = (r, \theta) \in Q = \{(r, \theta) | F(r, \theta) > c_0\} \quad (1.2) \]

has been studied in [1]. The multiple integration similar to the left hand side of (1.1) appears in many fields, such as the fundamentals of CT (computerized tomography)\(^2\), the electric

\(^1\) Department of Mechanics, South China University of Technology, Guangzhou 510641. P. R. China

\(^2\) The Project supported by the Natural Science Foundation of Guangdong Province of China
potential in an electric magnetic field\cite{1}, the analysis of electric chemistry\cite{4} and the contact problem\cite{5} etc. Since the solution shown in type (2.19) of \cite{1} holds only for its condition (2.11), i.e., \( \omega = F(r, \theta) \) is satisfied, this paper revises the errors in \cite{1} (such as (2.3), (2.4) and \( \omega \) undefining) and presents the general solution \( p \). Although the derivation and the result of the solution herein have no difference from the solution shown in type (2.10) of \cite{1}, detailed discussion on the solution’s conditions and properties, and the solution of the contact problem between rigid cone and elastic half space are given. The derivation of the latter is much simpler and clearer than that by Love (1939)\cite{9}, and can be used in calculating of material hardness-testing using instrument with conical punch.

II. Solution of Integral Equation (1.1) Subjected to Constraint (1.2)

The method of solution-derivation used herein is the same of \cite{1}, i.e., double integration of (1.1) is simplified to single integration by the theorem of integral mean value of function; functions of two variables \( p(r, \theta) \) and \( F(r, \theta) \) are reduced to that of one variable \( p(\omega) \), \( g(\omega) \) by means of isoplethic contour; and the solution is obtained from the reduced standard Abel integral equation by introducing Radon transform, transformation of variable. Of course one can also achieve this result without introducing Radon transform. The aim of introducing Radon transform is to emphasize the physical and geometrical concept of the method and to link up the CT field with the contact problem for further use. For convenience, the derived process is listed.

A Radon transform of function \( f \) of two variable \((s, \psi)\) in local polar coordinates is also a function of two variables \((h, \lambda)\) in global polar coordinates, which is defined\cite{9} by

\[
[Rf](h, \lambda) = \int_{-\infty}^{\infty} f(\sqrt{h^2+v^2}, \lambda+\arctan(v/h)) dv \quad (2.1)
\]

The Radon transform of \( f \) represents a line integration of \( f(s, \psi) \) along a straight line \( L \) with angle \( \psi \), and \( h \) indicates the distance between \( O(0, 0) \) and \( L, \lambda \) is the angle between \( OH \) and the \( x \)-axis. \( OH \perp L \), \( \overline{OH} = h \), and \( f(s, \psi) = 0 \) for \((s, \psi)\) locates outside the investigated region \( Q \) (Fig. 1).

![Fig. 1](image)

According to the definition of Radon transform of function, the left hand side (L. H. S.)