The Horizons of Two Schwarzschild Black Holes

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Abstract

The problem investigated is that of two Schwarzschild black holes of masses $m_1$ and $m_2 \ll m_1$, which are initially at rest on a time-symmetric spacelike hypersurface. An asymptotic formula as $m_2 \to 0$ is found for the maximum separation between $m_1$ and $m_2$ such that a connected component of the apparent horizon encloses both $m_1$ and $m_2$. The problem is solved numerically.

§(1): Introduction

The problem to be discussed is that of two Schwarzschild black holes of masses $m_1$ and $m_2 \ll m_1$. The initial condition is that there exists a time-symmetric spacelike hypersurface $\mathcal{L}$ on which $m_1$ and $m_2$ are at rest: the geometry of $\mathcal{L}$ is therefore axisymmetric. The object of the paper is to find the maximum separation between $m_1$ and $m_2$ for which there exists in $\mathcal{L}$ a connected component of the apparent horizon enclosing both $m_1$ and $m_2$. The result will be expressed in terms of an asymptotic formula as $m_2 \to 0$ [equation (10)].

The apparent horizon in $\mathcal{L}$ is calculated numerically. The basic theory is known [1-3] and numerical calculations have been carried out before [4], including some previous work by the author [5]. These papers considered the case $m_2 \approx m_1$ and the numerical procedures have to be reformulated to allow $m_2 \ll m_1$.

The apparent horizon has two important properties [6]: (a) in a stationary space-time the apparent horizon and the event horizon coincide; and (b) in general, if an apparent horizon exists there is a component of the event horizon outside it. Thus the maximum separation between $m_1$ and $m_2$ for which there exists
a connected component of the event horizon enclosing both \( m_1 \) and \( m_2 \) is greater than or equal to that of the apparent horizon calculated here.

§(2): Numerical Method

The theory of the location of the apparent horizon was discussed in Reference 5, Section 2, and it will be assumed that the reader is familiar with this. The essential features of the calculation are as follows. A time-symmetric spacelike hypersurface \( \mathcal{L} \) is considered. The geometry of \( \mathcal{L} \) is axisymmetric and therefore cylindrical coordinates \((\rho, \theta, z)\) are used. There are masses \( m_1 \) at the origin and \( m_2 \) at \( \rho = 0, z = a \). The metric in \( \mathcal{L} \) is

\[
\text{d}s^2 = \psi^4 (d\rho^2 + \rho^2 d\theta^2 + dz^2)
\]

where

\[
\psi = 1 + \frac{m_1}{2(\rho^2 + z^2)^{1/2}} + \frac{m_2}{2[\rho^2 + (z - a)^2]^{1/2}} \tag{1}
\]

A marginally trapped surface has cylindrical symmetry and may be viewed as a trajectory in the \((\rho, z)\) plane satisfying

\[
\ddot{z}^2 + \dot{\rho}^2 = Q^{-2} \tag{2}
\]

and

\[
(Q^2 \dot{z})' = Q^{-1} Q, z \\
(Q^2 \dot{\rho})' = Q^{-1} Q, \rho \tag{3}
\]

where \( Q = \rho \psi^4 \), \( \cdot' = d/d\lambda \) and \( \lambda \) is an affine parameter along the trajectory. If a trajectory starts and finishes at \( \rho = 0, z \neq 0 \) or \( a \), then the trajectory represents a marginally trapped surface. Trajectories with various differential initial values of \( z, z_0 \), are calculated and the marginally trapped surfaces are found by interpolation, as described in Ref. 5. The numerical integration of the equations (3) is straightforward when \( m_2/m_1 = O(1) \) (as is the case in Reference 5), but unfortunately there are problems here where \( m_2/m_1 \ll 1 \). The numerical error becomes apparent in the transition from one type of qualitative behaviour to another as \( z_0 \) is varied: it is no longer sharp (as theory requires), but occurs over a range. After some trial and error a better way of writing the equations was found.

Define \( \alpha \) by

\[
\dot{z} = Q^{-1} \cos \alpha \\
\dot{\rho} = Q^{-1} \sin \alpha \tag{4}
\]