EQUATIONS OF HYDRODYNAMICS IN GENERAL RELATIVITY
IN THE SLOW MOTION APPROXIMATION

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ABSTRACT

By means of a formal solution to the Einstein gravitational
field equations a slow motion expansion in inverse powers
of the speed of light is developed for the metric tensor.
The formal solution, which satisfies the deDonder coordi-
rate conditions and the Trautman outgoing radiation condi-
tion, is in the form of an integral equation which is sol-
volved iteratively. A stress-energy tensor appropriate to a
perfect fluid is assumed and all orders of the metric need-
ed to obtain the equations of motion and conserved quan-
tities to the 2\(\frac{1}{2}\) post-Newtonian approximation are found.
The results are compared to those obtained in another gauge
by S. Chandrasekhar. In addition, the relation of the
fast motion approximation to the slow motion approxima-
tion is examined.

§1: INTRODUCTION

In order to obtain successive approximations of the metric ten-
sor and the equations of motion for an arbitrary physical system,
the usual approach in general relativity is to assume a series ex-
ansion for the metric in powers of some smallness parameter and
then solve the Einstein field equations at each order of the expan-
sion. In order that these be a determinate set of equations, one
or another set of coordinate conditions must be imposed on the met-
ric. Because these coordinate conditions are used to modify the
field equations, it is necessary to check that solutions of the
modified equations satisfy these conditions and any assumed bound-
dary conditions—requirements increasingly more difficult to verify
as the order of the approximation increases.

If one assumes that matter moves non-relativistically, it is appropriate to use an expansion in inverse powers of the speed of light or more properly in powers of the small dimensionless ratio $\bar{v}/c$, where $\bar{v}$ is a typical velocity of the matter. This 'slow motion' approximation is particularly applicable to astrophysical systems and has been developed by numerous authors including Einstein, Infeld and Hoffmann in their theory for a system of 'point masses' (see Infeld and Plebański [8]) and in Fock's [8] post Newtonian method. For the case of a perfect fluid, the post Newtonian method reached fruition with the work of Chandrasekhar [8], Chandrasekhar and Nutku [5], and Chandrasekhar and Esposito [4].

In the series of papers by Chandrasekhar and his collaborators, the expansion is carried out to the $2^{3}$ post Newtonian approximation (pNA). In this approximation all terms up to $c^{-5}$ are included in the equations of motion. It is at this stage that gravitational radiation effects first appear, and in order to obtain them Chandrasekhar breaks with his previous papers and employs an ad hoc procedure. Up to this order he had used the standard post Newtonian method which, because the Sommerfeld outgoing radiation condition is not imposed, generates only the 'even' steps of the approximation (i.e. the $1, 2, 3, \ldots$ PNA which yield, in the equations of motion, highest orders of $2, 4, 6, \ldots$, respectively, in $1/c$). Radiative reaction can occur only in the 'odd' orders, since only these are sensitive to the direction of time. Unfortunately the basic assumption involved in a slow motion approximation, namely that $\bar{v}/c << 1$ and therefore derivatives with respect to $x_0 = ct$ raise orders of terms by one, implies that the PNA holds only in the near zone ($r << ct$). Thus to find solutions valid in the far zone, where the radiation condition is to be imposed, Chandrasekhar obtains modified field equations which specifically involve only the lowest order terms in the metric coefficients that derive from this boundary condition; using these he then falls back on the standard PNA to obtain the next odd terms in the series. In addition, these radiation terms are obtained with coordinate gauge conditions different from those used in his $1, 2$ and $3$ PNA, an awkwardness made permissible by the fact that the field equations never mix odd and even terms.

More recently, slow motion approximations have also been obtained via other schemes which still rely upon solving successive Poisson type differential equations at each order. Thus Persides [11] showed that by choosing independent coordinates $u, x, y, z$, where $u$ is the retarded time, the metric tensor can be expanded in powers of $c^{-1}$ with the $i$-th coefficients $A_{\mu \nu}^{(i)}$ determined from the previous ones by means of the field equations. The additional requirement that the coefficients of $c^{-n}$ can be expanded in powers of $r^{-1}$ in the radiation zone is used to specify more precisely the coordinate system (i.e. the gauge). On the other hand, Dixon [6] takes the line element in the form

$$ds^2 = e^{-2u}k_{ab}(dx^a + \xi^a dt)(dx^b + \xi^b dt) - \sigma^2 e^{2u} dt^2,$$