DETERMINING RELATIONS IN THE THEORY
OF DISSIPATIVE MEDIA

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There exist a number of ways of determining $\sigma - \dot{\varepsilon}$ relation ($\sigma$ and $\dot{\varepsilon}$ are six-dimensional vectors of stress and of strain rate) for a dissipative medium. In the theory of plasticity, the postulates of Il''yushin [1] and Draker [2] are used. For a more general class of media the hypothesis of potential [3, 4] and the thermodynamic principles of Ziegler [5] are widespread, but the results obtained in [3, 4] and those of [5] differ. Some authors, believing that the postulates accepted in [1-5] are insufficiently substantiated, use the dependences [6, 7] which are not subordinated to them. With a stricter approach, the $\sigma - \dot{\varepsilon}$ relation remains indeterminate [8, 9].

The present work, which is an extended exposition of the communication [10], presents a strict derivation of the $\sigma - \dot{\varepsilon}$ relation with sufficiently probable premises. The results are extended to media with memory.

1. Let $\sigma$ and $\dot{\varepsilon}$ be elements of an n-dimensional vector space $\mathbb{R}^n$. As a rule, a purely dissipative medium is specified by the dissipative function $D(\dot{\varepsilon}) = \sigma(\dot{\varepsilon}) \cdot \dot{\varepsilon} \Rightarrow 0$ [5], and therefore the problem of determining $\sigma(\dot{\varepsilon})$ amounts to the following: from the scalar product of two vectors and from one of them to find the other vector. We state the problem of determining $\dot{\varepsilon}(\sigma)$ from $D_0(\sigma) = \dot{\varepsilon}(\sigma) \cdot \sigma \Rightarrow 0$ analogously. It is understandable that for solving the one equation with n unknowns that we have, we must adopt additional probable premises.

We call $\sigma^*(\dot{\varepsilon})$ and $\dot{\varepsilon}^*(\sigma)$, for which $D(\dot{\varepsilon}) = \sigma^*(\dot{\varepsilon}) \cdot \dot{\varepsilon}$ and $D_0(\sigma) = \dot{\varepsilon}^*(\sigma) \cdot \sigma$, possible relations (or kinds of $\sigma - \dot{\varepsilon}$ relation) because they are possible solutions of the available equations. Among the possible relations there are real (true) ones for whose determination we formulate additional conditions.

I. The values $M$ assumed by the dissipative function $D(\dot{\varepsilon})$ determine fully the dependence $\dot{\varepsilon}(\sigma)$, with specified $\sigma$ nothing except the scalars $M = D(\dot{\varepsilon})$ can be used for selecting $\dot{\varepsilon}$ from $\dot{\varepsilon}^*$.

II. The relation $\dot{\varepsilon}(\sigma)$ is single-valued for all $\sigma \in \mathbb{R}^n$.

We solve the problem. We fix $\sigma$ arbitrarily and, subsequently, some hitherto unknown value $M = D_0(\sigma) = D(\dot{\varepsilon})$ ($D_0(\sigma)$ is unknown but it exists because $\dot{\varepsilon}(\sigma)$ is single-valued), and we seek the vector $\dot{\varepsilon}$ corresponding to them. The ends of the possible vectors $\dot{\varepsilon}^*$ have to belong to the surface $D(\dot{\varepsilon}) = M$ and, by definition, to the plane $\sigma \cdot \dot{\varepsilon}^* = M$. If these surfaces intersect, then there is no way to select one vector $\dot{\varepsilon}$ from $\dot{\varepsilon}^*$ because for that, in addition to $\sigma$ and $M$, additional conditions would have to be used. Only in cases of contact at one point, i.e., if the plane $\sigma \cdot \dot{\varepsilon}^* = M$ is the plane of support of the surface $D(\dot{\varepsilon}) = M$, is the vector $\dot{\varepsilon}$ determined univalently (Fig. 1). It follows from the contact that the dissipative function $D(\dot{\varepsilon})$, for $k = \dot{\varepsilon}/\dot{\varepsilon} = \text{const}$, increases, $D(\dot{\varepsilon}) = M$ is a convex surface, and

$$\sigma \cdot \dot{\varepsilon} \geq \sigma^* \cdot \dot{\varepsilon}^* \quad \text{for} \quad D(\dot{\varepsilon}) \geq D(\dot{\varepsilon}^*).$$

We call the extremal principle (1) the principle of the maximum scalar product. Solving the problem for the conditional extremum, we obtain from this principle for the smooth functions $D(\dot{\varepsilon})$

$$\sigma = \lambda \nabla_\varepsilon \cdot D; \quad \lambda = (\nabla_\varepsilon \cdot D \cdot \dot{\varepsilon})^{-1} D,$$

where $\nabla$ is the operator of the gradient. Thus the $\sigma - \dot{\varepsilon}$ relation is determined. Instead of varying the scalar $M$, the modulus $\sigma = (\sigma \cdot \sigma)^{1/2}$ with fixed $M$ and $n = \sigma/\sigma$ may be varied. If we take the vectors $\sigma$ and $\sigma^*$, where $D_0(\sigma) = D_0(\sigma^*)$, and $\dot{\varepsilon}$ and $\dot{\varepsilon}^*$ corresponding to them, we have $\sigma \cdot \dot{\varepsilon} = \sigma^* \cdot \dot{\varepsilon}^*$, or

$$\sigma \cdot \dot{\varepsilon} \geq \sigma^* \cdot \dot{\varepsilon}^* \quad \text{for} \quad D_0(\sigma) \geq D_0(\sigma^*);$$

$\dot{\varepsilon} = \nabla_\sigma D_0; \quad \n = (\nabla_\sigma D_0 \cdot \sigma)^{-1} D_0.$

The relation $\dot{e}(\sigma)$ is found; the principle of maximum is correct in this case, too. We note that formally the conditions I and II can be expressed as follows: the dependence $\dot{e}(\sigma)$, determined from the equations $\dot{e}(\sigma) \cdot \sigma = M$ and $D(e) = M$, is single-valued for all $\sigma \in R^3$.

We will generalize some results. If $D = D(\xi, \omega)$, where $\omega$ is the set of parameters depending both on the instantaneous state of the system and on the history of its deformation, then the principle (1) is proved analogously with fixed values of $\omega$.

Condition II can be weakened by admitting the existence of nonconcavity on the surface $D(\xi) = M$: if $\dot{e}(\sigma)$ is not single-valued, then for these $\dot{e}$ the dependence $\sigma(\dot{e})$ is single-valued. In fact, in case of ambiguity for $M$ and $\sigma$, it is possible that the surface $D(\xi) = M$ and the surface $\dot{e}^* \cdot \sigma = M$ intersect, or that they touch along a flat section. At any point of intersection of the mentioned surfaces there is the possibility of contact between $D(\xi) = M$ and the plane corresponding to the other $\sigma$, i.e., at least two vectors $\sigma$ will correspond to this vector $\dot{e}$. Only in case of contact along the plane section is the dependence $\sigma(\dot{e})$ single-valued; this case is also described by principle (1).

To the plane section on the surface $D(\xi)$ corresponds the singularity on the surface $D_0(\sigma)$. If this singularity is formed by the intersection of $r$ smooth surfaces $D_{0i}(\sigma)$, $r = 2, 3, \ldots, \infty$, then it follows from (3) that

$$\dot{e} = v_i V_{\sigma} D_{0i}; \quad v_i V_{\sigma} D_{0i} \cdot \sigma = D_0.$$  \hfill (5)

For determining the dependence $\sigma(\dot{e})$ in more general cases (mutual ambiguity of the $\sigma - \dot{e}$ relation, existence of concave sections on $D(\xi) = M$, etc.), we adopt a condition which we call the condition of single principle.

III. The principle determining $\sigma(\dot{e})$ is invariant with respect to a change in the kind of the function $D(\xi)$, i.e., it does not depend on the actual form of $D(\xi)$.

If the function $D(\xi)$ is changed in such a way that the dependence $\dot{e}(\sigma)$ is single-valued, then principle (1) must be fulfilled, which may be written in the form

$$\delta (\sigma \cdot \dot{e}) = 0 \text{ for } \delta D(\dot{e}) = 0, \delta \sigma = 0.$$ \hfill (6)

Therefore, the general principle, correct for any function $D(\dot{e})$, must contain the principle (6) as a special case; however, since nothing, except $D(\dot{e})$, affects the dependence $\sigma(\dot{e})$, this general principle coincides with (6). It remains to demonstrate that not all steady-state values of (6) have to be maximal, as in (1). In fact, in case of contact of the surface $D(\xi) = M$ and the plane $\dot{e}^* \cdot \sigma = M$ on the concave section or in case of bending of $D(\xi) = M$, the steady-state values are not maximal, but for these $\dot{e}$ the dependence $\sigma(\dot{e})$ exists because the function $D(\dot{e})$ exists.

Since $\delta D = \sigma \cdot \delta \dot{e} + \varepsilon \cdot \delta \sigma = \delta D_0$, we obtain from the principle (6) that

$$\delta (\sigma \cdot \dot{e}) = 0 \text{ for } \delta D_0(\sigma) = 0, \delta \dot{e} = 0.$$ \hfill (7)

Principles (6) and (7) can be merged into one which we will call the principle of the steady-state scalar product:

$$\delta (\sigma \cdot \dot{e}) = 0 \text{ for } \delta D(\dot{e}) = 0, \delta D_0(\sigma) = 0.$$ \hfill (8)