Solutions of Five-Dimensional General Relativity without Spatial Symmetry

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Conformastationary solutions of the five-dimensional vacuum Einstein equations, depending on one or two harmonic potentials, are constructed. The solutions depending on one potential fall in three distinct classes. Solutions of two of these classes may be combined to yield a class of solutions depending on two potentials, which correspond to the Israel–Wilson–Perjès solutions of the Einstein–Maxwell theory. The asymptotically flat solutions of this class describe systems of rotating electric or magnetic monopoles.

1. INTRODUCTION

Five-dimensional general relativity with a compact fifth dimension, also commonly known as Kaluza–Klein theory [1], is a unified field theory of gravitation and electromagnetism [2]. The extent to which classical solutions of five-dimensional general relativity can be related to those of the four-dimensional Einstein–Maxwell theory has been discussed in [3], where it was shown that a precise correspondence is expected for stationary solutions depending on two real potentials. This correspondence was established in the case of axisymmetric “vacuum” solutions [4] (in the classification of [5], the class $\mathcal{E} = -1$), and of electrostatic, or magnetostatic, solutions (the class $\mathcal{E} = \mathcal{E}^*, \psi = \psi^*$); it was also shown that to the class $\mathcal{E} = +1$ of the stationary axisymmetric solutions of the Einstein–Maxwell theory, there corresponds a class of regular solutions of five-dimensional general relativity.

The remaining class of stationary Einstein–Maxwell fields depending on two real potentials is the exceptional class $\mathcal{E} = 0$ of conformastationary
solutions [5]. The spatial sections of the corresponding metrics are conformally flat, so that solutions without spatial symmetry, generalizing the static Papapetrou–Majumdar solutions [6], can be constructed from a complex harmonic function [7, 8].

The purpose of this paper is to investigate whether such conformastationary solutions also exist in the five-dimensional theory. In the second section we study the stationary solutions of the five-dimensional Einstein equations with conformally flat spatial sections and depending on a single real potential. These solutions fall in three distinct classes. The solutions of class (a), which include asymptotically flat electrostatic as well as dyon solutions, are regular. The asymptotically flat solutions of class (b) are dyon solutions with equal electric and magnetic charges [9]. The class (c), which corresponds to the four-dimensional Papapetrou–Majumdar class, includes asymptotically flat electrostatic and magnetostatic solutions, these latter being the well-known multimonompole solutions [10, 11].

In the third section, we show how to combine solutions of classes (b) and (c) so as to obtain a class of solutions depending on two real harmonic functions, which corresponds to the Israel–Wilson–Perjès class [7, 8]. The asymptotically flat solutions of this class describe systems of noninteracting, rotating electric or magnetic monopoles. The corresponding five-dimensional metrics are given explicitly in the one-monopole, axisymmetric case. These metrics have a nonremovable ring singularity.

2. SOLUTIONS DEPENDING ON ONE POTENTIAL

In the Maison formalism [12], the reduction of the 2-stationary five-dimensional Einstein equations (written in the case of two commuting Killing vectors, one of which is time-like) to three-dimensional equations is achieved in the following way. We first decompose the five-dimensional metric tensor $g_{\mu \nu}$ $(\mu = 1, \ldots, 5)$, assumed to be of signature $(- - - + - )$, into the tensors

$$\begin{align*}
\lambda_{ab} &\equiv g_{ab} \\
\mathcal{A}^a_i &\equiv \lambda^{ab} g_{ib} \\
h_{ij} &\equiv \tau (g_{ij} - \lambda_{ab} \mathcal{A}^a_i \mathcal{A}^b_j)
\end{align*}$$

(1)

$(i = 1, 2, 3; a = 4, 5)$, where $(\lambda^{ab}) = (\lambda_{ab})^{-1}$, and $\tau = -\det \lambda_{ab}$. We introduce the twist 2-vector $\omega_a$, such that

$$\omega_{a,i} = h^{-1/2} \tau \lambda_{ab} h_{il} \eta^{lk} \mathcal{A}^b_j$$

(2)