Potential Surfaces for Time-like Geodesics in the Curzon Metric

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In this paper we deduce the general pattern of the potential surfaces for time-like geodesics in the Curzon metric. We find that for fairly small energies and orbital angular momenta, the time-like geodesics group into two sets; the geodesics of one set tend to the z-axis as $R = (r^2 + z^2)^{1/2} \to 0$, $R = 0$ being a directional singularity, the others tend to the r-axis. At low energies these two sets are detached but they merge together as the energy increases. Stable circular motion is confined to the $z = 0$-plane and an energy threshold for stationary motion exists and is equal (per unit of rest-mass energy) to $\approx 0.945$, a value almost indistinguishable from that in the Schwarzschild space-time.

1. INTRODUCTION

The Curzon solution [1,2] describes a static and axisymmetric vacuum space-time whose line element, written in Weyl’s canonical coordinates and in units such that $c = 1 = G$ (Ref. 3, Ch.18, Section 1), is given by:

$$ds^2 = -e^{2\lambda}dt^2 + e^{2\nu-2\lambda}(dr^2 + dz^2) + r^2e^{-2\lambda}d\phi^2. \quad (1)$$

$$\lambda = -\frac{m}{R}; \quad \nu = -\frac{m^2r^2}{2R^4}; \quad R = (r^2 + z^2)^{1/2}.$$ 

This solution was studied in some detail after it was found that the singularity at $R = 0$ had directional properties. In fact as $R \to 0$, the first
Kretschmann invariant $\mathcal{K} = R^{ijkl}R_{ijkl}$ tends to values which can be zero, infinite or finite, according to the direction of approach to the singularity. The paths to the singularity which were considered to monitor the behaviour of $\mathcal{K}$ were the geodesics of the metric (1). They have been investigated mainly in their asymptotic behaviour as $R \to 0$ [4-6]. Introducing a suitable coordinate system, the singularity was found to possess a ring-like structure which, to some extent, clarifies its directional behaviour. In fact not all the geodesics which tend to $R = 0$ hit the singularity; some of them go across the ring and reach a point of internal infinity.

The rather complicated and unexpected structure of the Curzon singularity forces one to search for other interesting properties of this space-time solution. In this paper we study the time-like geodesics and obtain a complete classification with respect to the constants of the motion considered as free parameters. This task is accomplished by deriving the potential surfaces for the geodesic motion and, as is well known, these are recognized after a first integration of the geodesic equations. This is done analytically in Section 2. In our case, the symmetries of metric (1), and the non separability of the equations of motion in the $r$ and $z$ coordinates, enable one to determine only the full turning points of the geodesics, namely the points where the $r$ and $z$ components of the tangent vector (namely $\dot{r}$ and $\dot{z}$) vanish simultaneously. As a consequence, in the $(r-z)$-plane, the corresponding potential surfaces can only fix the coordinate ranges where the geodesics are confined. Evidently the shape of the potential surfaces will not tell us whether a geodesic actually strikes a turning point or not; to know this one needs to integrate the geodesic equations once more, a task which requires a numerical approach. The result of this operation would be a detailed description of a single trajectory (or class of them; see Ref. 6), while the effect of the present analysis is to identify at once, for each set of constants of the motion, all the permitted classes of time-like geodesics. The special class of time-like (spatially) circular geodesics is studied in Section 3 and an energy threshold for stationary motion is deduced which barely differs from that of the Schwarzschild solution.

Finally in Section 4 we study the properties of the Kretschmann invariant in the Curzon solution to verify the consistency of a suggestion according to which that invariant is related to peculiar properties of the space-time. In recent papers, in fact [7,8], it was shown that in some exact solutions like the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman solutions, a remarkable correspondence exists between the vanishing of the Kretschmann invariant at space-time points where the curvature tensor is finite and non zero and the existence of a gravitationally repulsive domain. It was then conjectured that this correspondence could be a