METHOD OF SOLVING THREE-DIMENSIONAL NONLINEAR DYNAMIC PROBLEMS FOR COMPOUND BODIES OF ROTATION

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A mathematical formulation is given for the three-dimensional nonstationary geometrically linear state of stress and strain in a compound body of rotation in a physically nonlinear setting. This incorporates failure and plasticity in the material. The method is based on expanding the unknowns as Fourier series and applying a numerical technique, which is shown by numerical experiments to be suitable for treatments in this class.

Recently, there has been increased interest in calculations on states of stress and strain in structures subject to nonstationary loading. An adequate description can be given from the mechanics of continuous media. Sometimes, one makes reasonable assumptions to formulate the treatment with satisfactory accuracy in a one-dimensional or two-dimensional setting. However, there are situations where the geometrical characteristics and/or the load are such that only three-dimensional equations can be used. An analytic solution is usually not available for a three-dimensional dynamic boundary-value treatment in partial derivatives, so numerical methods of integration are widely used. These can be divided into three groups.

1. Direct numerical integration of the partial differential equations, e.g., by finite-difference techniques on a three-dimensional net. Typical examples are the Wilkins three-dimensional algorithm [1] and modified forms of it [2, 3]. These methods enable one to incorporate nonlinear effects fairly readily, which are primarily associated with failure, physical and geometrical nonlinearities, and so on, but this requires large memory volumes and long run times.

2. An assumption is made about the distribution of the unknowns along some spatial coordinate. For example, in [4] a method was given for numerical examination of the three-dimensional dynamics of a thick-walled cylindrical shell based on simulating the initial structure as a set of thin-walled shells in conjunction with a piecewise-linear distribution for the displacements in the multilayer pocket. Such methods do not incorporate accurately the propagation of waves along the coordinate on which a constraint is imposed on the function behavior (the thickness in the [4] case).

3. All the unknown and known variables are represented as orthogonal-function series along one or more coordinates, with the use of a numerical method. One often uses Legendre-polynomial expansions [5], Fourier series [6, 7], and so on. Such methods are usually called numerical-analytic. They are more economical than algorithms of the first type and more accurate than ones of the second, but they have certain limitations associated primarily with the shape of the structure and the equation nonlinearity.

To examine the response of a body of rotation of pulses, one uses \( r, \varphi, z \) cylindrical coordinates with a method based on Fourier expansion with respect to the angular coordinate \( \varphi \). The traditional application is to linear treatments \([6, 7]\). That method has been used with a class of nonlinear three-dimensional treatments for nonstationary gas-hydroelasticity \([8]\).

Here I give a similar method for a three-dimensional dynamic geometrically linear state of stress and strain in a compound body of rotation in a physically nonlinear setting: this incorporates the failure or plasticity. The essence of the method is demonstrated on an example.

**Object and Mathematical Formulation.** Consider a thick-walled two-layer cylinder with finite length \( 2L \) in an \( r, \varphi, z \) cylindrical coordinate system. The ends of the cylinder \( (z = \pm L) \) are rigidly clamped, and the internal side surface \( (r = R_1) \) is free of load, while at the outer surface \( (r = R_2) \), a pressure pulse \( P(z, \varphi, t) \) begins to act at the initial instant \( t = 0 \):

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in which

\[
q(t) = \begin{cases} 
\frac{P_d}{T}, & 0 \leq t \leq T; \\
\frac{P_d(2 - t/T)}{T}, & T \leq t \leq 2T; \\
0, & t \geq 2T.
\end{cases}
\]

One can restrict consideration to a quarter of the cylinder \((0 \leq z \leq L, 0 \leq \varphi \leq \pi)\) with the use of symmetrical boundary conditions in the planes \(z = 0\) and \(\varphi = 0, \pi\).

We write the formulation in cylindrical coordinates. At all the internal points in a layer, we have the equations of motion

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} &= \rho \ddot{u}_r; \\
\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial \varphi} + \frac{\partial \sigma_{\varphi\varphi}}{\partial z} + \frac{2\sigma_{rr}}{r} &= \rho \ddot{u}_\varphi; \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= \rho \ddot{u}_z.
\end{align*}
\]

Here \(\sigma_{\alpha\beta}\) are the stress tensor components, \(u_\alpha\) the displacement velocity vector \((\alpha, \beta = r, \varphi, z)\), and \(\rho\) the density of the material; a dot above a parameter denotes differentiation with respect to time \(t\).

The Cauchy relations are

\[
\begin{align*}
e_{rr} &= \partial u_r/\partial r; \quad e_{r\varphi} = u_r/r + \partial u_\varphi/\partial \varphi; \\
e_{zz} &= \partial u_z/\partial z; \quad e_{rz} = \partial u_r/\partial z + \partial u_z/\partial r; \\
e_{\varphi\varphi} &= \partial u_z/r^2 + \partial u_\varphi/\partial \varphi; \quad e_{\varphi r} = \partial u_r/r^2 + \partial u_\varphi/\partial r - u_\varphi/r,
\end{align*}
\]

in which \(e_{\alpha\beta}\) are the components of the strain rate tensor \((\alpha, \beta = r, \varphi, z)\).

We consider the physical equations in more detail. Two models are considered for the behavior of the layer materials under nonstationary loading.

**I. Model for Elastically Failing Material (Model I).** We assume that an elementary volume can be in one of four states [9]:

1) defect-free material, isotropic, elastic, and obeying Hooke’s law;

2) there is a crack lying in a plane with normal \(n\). There are no stresses \(\sigma_{nn} > 0\) in that microvolume or tangential stresses at the surfaces of the crack;

3) two cracks in the element in two mutually perpendicular planes having normals \(n\) and \(m\) correspondingly. This means that there are no stresses \(\sigma_{nn} > 0\) and \(\sigma_{mm} > 0\) and no tangential stresses in the planes of the cracks, i.e., the stress tensor is diagonal in the coordinate system related to these normals; and

4) a microvolume contains three mutually perpendicular cracks in planes with normals \(n\), \(m\), and \(l\). There are no tangential stresses in the planes of the cracks, and the normal ones can only be compressive.

It is assumed that the cracks in the material can arise only in certain planes: \(r = \text{const}, \varphi = \text{const},\) or \(z = \text{const}\). The surface \(r = \text{const}\) is strictly speaking cylindrical, not planar, but we neglect the crack curvature within the elementary volume.

The stress state at each point is characterized not only by the stress tensor but also by a certain integer-valued damage function \(J(r, z, \varphi, t)\), which can take the following values:

- \(J = 0\) \(\Rightarrow\) material in state 1;
- \(J = 1\) \(\Rightarrow\) state 2, crack in the plane \(r = \text{const}\);
- \(J = 2\) \(\Rightarrow\) state 2, crack in plane \(z = \text{const}\);
- \(J = 3\) \(\Rightarrow\) state 2, crack in plane \(\varphi = \text{const}\);
- \(J = 4\) \(\Rightarrow\) state 3, two cracks in planes \(r = \text{const}\) and \(z = \text{const}\);