Determinant Formulas with Applications to Designing When the Observations Are Correlated*

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Abstract. In the general linear model consider the designing problem for the Gauß-Markov estimator or for the least squares estimator when the observations are correlated. Determinant formulas are proved being useful for the D-criterion. They allow, for example, a (nearly) elementary proof and a generalization of recent results for an important linear model with multiple response. In the second part of the paper the determinant formulas are used for deriving lower bounds for the efficiency of a design. These bounds are applied in examples for tridiagonal covariance matrices. For these examples maximin designs are determined.

Key words and phrases: Determinant formula, general linear model, correlated observations, D-criterion, efficiency of designs, linear model with multiple response, lower bounds for the efficiency, tridiagonal matrices as covariance structure, maximin designs.

1. Introduction, notations and preliminary results

Consider a general linear model

\[ Y = X\beta + Z \]

where \( X \) is a known real \((n \times m)\)-matrix, \( n \geq m \), \( \beta \in \mathbb{R}^m \) is an unknown parameter vector and \( Z \) is an \( n \)-dimensional real random vector with

\[ \text{E} Z = 0_n = (0, \ldots, 0)^\top \in \mathbb{R}^n, \quad \text{Cov} \ Z = C \text{ positive definite}. \]

In this paper we are interested in estimating \( \beta \). In the first instance we assume that \( C \) is known. For \( \beta \) being estimable \( X \) must be of full rank, that is, \( \text{rank} (X) = m \). We consider two estimators mostly used for estimating \( \beta \). Firstly the best linear unbiased estimator, the so-called Gauß-Markov estimator

\[ (X^\top C^{-1}X)^{-1}X^\top C^{-1} : \mathbb{R}^n \to \mathbb{R}^m \]

* Parts of the paper are based on a part of the author’s Habilitationsschrift Bischoff (1993a).
having covariance matrix \((X^\top C^{-1}X)^{-1}\), secondly the ordinary least squares estimator

\[(X^\top X)^{-1}X^\top : \mathbb{R}^n \to \mathbb{R}^m,\]

having covariance matrix \((X^\top X)^{-1}X^\top CX(X^\top X)^{-1}\). It is well-known that range(CX) = range(X) is a necessary and sufficient condition such that the Gauss-Markov estimator and the least squares estimator coincide; see for example Zyskind (1967) or Kruskal (1968), see also Rao (1967). Note, we do not distinguish between a linear mapping from \(\mathbb{R}^d\) to \(\mathbb{R}^j\) and its unique matrix representation with respect to the standard bases of unit vectors.

If designing for the general linear model is of interest the model matrix \(X\) is determined by choosing an (exact) design \(\tau \in \mathcal{E}^n\) where \(\mathcal{E}\) is the experimental region. To emphasize the dependence on \(\tau\) we write \(X_\tau\). Then the class \(\mathcal{E}_n\) of feasible designs for estimating \(\beta\) is given by \(\mathcal{E}_n = \{\tau \in \mathcal{E}^n : \text{rank}(X_\tau) = m\}\). So, for \(\tau\) ranging over \(\mathcal{E}_n\), we have a class of linear models which we denote by \(LM(X_\tau, C : \tau \in \mathcal{E}_n)\).

Almost all optimality criteria in design theory depend on the covariance matrix of the Gauss-Markov estimator. An optimal design \(\tau \in \mathcal{E}_n\) minimizes an appropriate functional of the corresponding covariance matrix or equivalently maximizes an appropriate functional of the inverse of the covariance matrix; see Pukelsheim (1993), Sections 5 and 6). One of the more commonly used criteria for choosing a design \(\tau \in \mathcal{E}_n\) is the famous D-optimality criterion. A design \(\tau^* \in \mathcal{E}_n\) is called D-optimal (for \(LM(X_\tau, C : \tau \in \mathcal{E}_n)\)) if

\[\det(X_\tau^T C^{-1}X_\tau) \leq \det(X_{\tau^*}^T C^{-1}X_{\tau^*})\]

for all \(\tau \in \mathcal{E}_n\). The statistical aim of the paper is to give a general concept how one can tackle the problem of finding optimal or efficient designs with respect to the D-criterion when observations are correlated.

For special factorial linear models with correlated observations exact optimal designs are known; see for example, Kiefer and Wynn (1983, 1984), Budde (1984), Kunert and Martin (1987), and the references cited there.

Only little is known on optimal designs of regression models when the observations are correlated; see Bischoff (1992) and the references cited there. Recently, Bischoff (1992, 1993b) has stated conditions such that a D-optimal design for uncorrelated observations with common variance is also D-optimal for correlated observations.

Because it is difficult to determine optimal designs for a linear model with correlated observations, a hybrid approach is often chosen: namely to look for an optimal design not in the class of all possible designs \(\mathcal{E}_n\) but only in the class of all designs which are optimal for the uncorrelated case with common variance; see Kiefer and Wynn (1981) and the literature cited there, see also Budde (1984). We explain the above approach in more detail because it may be used in two different ways. However, we consider these two approaches for the D-criterion only. To this end let the class of linear models \(LM(X_\tau, C : \tau \in \mathcal{E}_n)\) be given.

Firstly an optimal design \(\tau_0 \in \mathcal{E}_n\) in the above sense minimizes the determinant of the covariance matrix of the ordinary least squares estimator in the class of all