Diagonalization of Compact Operators in Hilbert Modules over Finite $W^*$-Algebras

V.M. Manuilov

Abstract: It is known that a continuous family of compact self-adjoint operators can be diagonalized pointwise. One can consider this fact as a possibility of diagonalization of the compact operators on Hilbert modules over a commutative $W^*$-algebra. The aim of the present paper is to generalize this fact to a finite $W^*$-algebra $A$ not necessarily commutative.

We prove that for a compact operator $K$ acting on the right Hilbert $A$-module $H_A^*$ dual to $H_A$ under slight restrictions one can find a set of "eigenvectors" $x_i \in H_A^*$ and a non-increasing sequence of "eigenvalues" $\lambda_i \in A$ such that $K x_i = x_i \lambda_i$ and the selfdual Hilbert $A$-module generated by these "eigenvectors" is the whole $H_A^*$. As an application we consider the Schrödinger operator in a magnetic field with irrational magnetic flow as an operator acting on a Hilbert module over the irrational rotation algebra $A_\theta$ and discuss the possibility of its diagonalization.

Key words: Diagonalization of operators, Hilbert module, compact operator, $W^*$-algebras
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1. Introduction

The classical Hilbert-Schmidt theorem states that any compact self-adjoint operator acting on a Hilbert space can be diagonalized. It is also known that a continuous family of compact operators is diagonalizable. When active study of Hilbert modules began some results were obtained concerning diagonalizability of some operators acting on these modules. R.V. Kadison ([8], [9]) proved that a self-adjoint operator in a free finitely generated module over a $W^*$-algebra is diagonalizable. Later on some other interesting results about diagonalization of operators appeared [7], [16], [24]. This paper is a step in the same direction and is concerned with the diagonalization of compact operators in the Hilbert module $H_A^*$ over a finite $W^*$-algebra $A$. Its main results were announced in [15].

The present paper is organized as follows:

In Section 2 we study some properties of Hilbert modules over finite $W^*$-algebras related with orthogonal complementability. The main technical result is the isomorphy of $H_A^*$ and the orthogonal complement to $A$ in $H_A^*$. In Section 3 we recall the basic facts about the compact operators in Hilbert modules. Here we also give an example showing that the module $H_A$ is not sufficient to diagonalize compact operators, so we must turn to its dual module $H_A^*$. Section 4 contains the proof of the main theorem of this paper about diagonalization of a compact operator in the
module $H^*_A$. Here we also discuss the uniqueness condition for the “eigenvalues” of
this operator. Section 5 deals with quadratic forms on Hilbert modules related to a
self-adjoint operator. Properties of these forms are mostly the same as on a Hilbert
space. In Section 6 we discuss an example which motivated the present paper. We
consider the perturbated Schrödinger operator with irrational magnetic flow as an
operator acting on a Hilbert module over the irrational rotation algebra $A_\theta$ and we
show that this operator is diagonalizable.

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2. Orthogonal Complements in Hilbert Modules over Finite
$W^*$-Algebras

Throughout this paper $A$ is a finite $W^*$-algebra admitting a central decomposition
into a direct integral over a compact Borel space. By $\tau$ we denote a normal faithful
finite trace on $A$ with $\tau(1) = 1$. Recall some facts about Hilbert modules. Standard
references on them are [10], [12], [19]. If $B$ is a $C^*$-algebra, we denote by $H_B$ (another
usual denotation is $l_2(B)$) the right Hilbert $B$-module consisting of the sequences
$(x_i), x_i \in B, \ i \in \mathbb{N}$ for which the series $\sum_i x_i^* x_i$ converges in the norm topology
of $B$ with the inner product $\langle x, y \rangle = \sum_i x_i^* y_i$ and the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. Let
$H_B^*$ be its dual module, $H_B^* = \text{Hom}_B(H_B; B)$. It is shown in [20] that in the case
of $W^*$-algebras the inner product on the module $H_B$ can be extended to the inner
product on the module $H_B^*$ and this module is selfdual, i.e., $(H_B^*)^* = H_B^*$.

Let $M \subset H_B^*$ be a Hilbert $B$-submodule. By $M^\perp$ we denote its orthogonal com-
plement in $H_B^*$. It is well-known ([3]) that if $M$ is a finitely generated projective
Hilbert $B$-submodule in $H_B^*$, then it is orthogonally complemented: $H_B^* = M \oplus M^\perp$.
If we replace $H_B^*$ by $H_B$, then the orthogonal complement to $M$ in $H_B$ is isomor-
phic to $H_B$, but it is not known in general whether $M^\perp$ and $H_B$ are isomorphic.
The following theorem solves this problem in the case of modules over a $W^*$-algebra
decomposable into a direct integral of finite factors and having a faithful finite trace.

Theorem 2.1. If $M$ is a finitely generated projective $A$-submodule in $H_A^*$, then
$M^\perp$ is isomorphic to $H_A^*$.

Proof. The idea of the following proof is contained in [3]. Let $g_1, \ldots, g_n$ be gener-
ators of the module $M$. Without loss of generality we can assume that the operators
$\langle g_i, g_i \rangle \in A$ are projections, $\langle g_i, g_i \rangle = p_i$. Let $\{e_m\}$ be the standard basis of the
module $H_A \subset H_A^*$. Fix $\varepsilon < 0$ and define elements $e_m' \in M^\perp$ by the equality
$$e_m' = e_m - \sum_{i=1}^n g_i(g_i, e_m).$$