ON NONPARAMETRIC TESTS FOR SYMMETRY IN $R^m$

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Abstract. This paper considers the problem for testing symmetry of a distribution in $R^m$ based on the empirical distribution function. Limit theorems which play important roles for investigating asymptotic behavior of such tests are obtained. The limit processes of the theorems are multiparameter Wiener process. Based on the limit theorems, nonparametric tests are proposed whose asymptotic distributions are functionals of a multiparameter standard Wiener process. The tests are compared asymptotically with each other in the sense of Bahadur.

Key words and phrases: Asymptotic distribution, test for symmetry, $L_1$-norm, $L_2$-norm, empirical process, central limit theorems, goodness-of-fit tests, multiparameter Wiener process, density estimator, approximate Bahadur efficiency.

1. Introduction

In this paper, we deal with the problem of testing symmetry of a distribution in $R^m$. Throughout the paper, we assume the center of symmetry is known. Hence, without loss of generality, we consider the problem for testing symmetry about 0.

In matched pair treatment effect experiments with $m$ measurements on each member of $n$ pairs (each pair has one treated and one control member), the natural null-hypothesis is symmetry about the zero vector. In this case, the difference “treatment-control” of the responses in each pair would be the basic $X$ vectors and under the null-hypothesis of no treatment effect, symmetry holds. Here a “stochastically larger than symmetry” alternative is called for.

In the 1-dimensional case, many statistics have been proposed for the goodness-of-fit test for symmetry. For example, we can mention Butler (1969), Rothman and Woodroofe (1972), Shorack and Wellner ((1986), Section 22), Aki (1987), Csörgő and Heathcote (1987) and Aki and Kashiwagi (1989).

Let $X = (X_1, X_2, \ldots, X_m)'$ be a random vector in $R^m$ which is symmetric about 0. Suppose that $F_1, F_2, \ldots, F_m$ are 1-dimensional cumulative distribution functions which are symmetric about 0. Then the distribution of $(F_1(X_1), F_2(X_2), \ldots, F_m(X_m))'$ is symmetric about $(1/2, 1/2, \ldots, 1/2)'$. This can be seen as follows:
The symmetry of $X$ and $F_i$'s imply respectively that

$$(X_1, X_2, \ldots, X_m)' \overset{d}{=} (-X_1, -X_2, \ldots, -X_m)'$$

and for every $x \in \mathbb{R}$, $F_i(x) + F_i(-x) = 1, i = 1, 2, \ldots, m$. Therefore, for each $x_1, x_2, \ldots, x_m \in \mathbb{R}$, it holds that

$$P(1 - F_1(x_1) \leq x_1, \ldots, 1 - F_m(x_m) \leq x_m)$$

$$= P(1 - F_1(-x_1) \leq x_1, \ldots, 1 - F_m(-x_m) \leq x_m)$$

$$= P(F_1(x_1) \leq x_1, \ldots, F_m(x_m) \leq x_m).$$

In the above statement, if we take $F_1, F_2, \ldots, F_m$ so as to be continuous and strictly increasing, every continuous and symmetric distribution about $0$ in $\mathbb{R}^m$ is transformed to a continuous and symmetric distribution about $(1/2, 1/2, \ldots, 1/2)'$ in $[0, 1]^m$. Consequently, the problem of investigating the symmetry of a distribution in $\mathbb{R}^m$ can be reduced to that of investigating the symmetry of the transformed distribution in $[0, 1]^m$ as far as we know the center of symmetry.

Aki (1987) proposed a limit theorem which plays an important role for deriving asymptotic distributions of tests for symmetry based on the empirical distribution function. In Section 2, we give a limit theorem which can be regarded as the $m$-dimensional analogue of the limit theorem. The limiting process of the theorem is a $(m$-parameter) Wiener process w.r.t. a $(m$-dimensional) distribution function. Next, we consider a transformation of the process to a $(m$-parameter) standard Wiener process by using the idea of Khmaladze (1988). Further, these results are extended in a general framework on which the central limit theorem for empirical processes indexed by uniformly bounded families of functions is studied recently (cf. Giné and Zinn (1984, 1986)). In Section 3, an integral test is proposed whose asymptotic distribution is the $L_2$-norm of a (multiparameter) standard Wiener process. The distribution is also investigated. The test is not coordinate free (see Remark 3.1 below). Hence, the test procedure is not invariant under the rotations of the data without extra considerations. Some statisticians may regard it as a defect. However, if we always rotate the data to a following data-dependent coordinate system before transforming them into the unit cube, then we can make the test procedure to be invariant under the rotations. For example, consider the following rotation of the data: Let $X_1, \ldots, X_n$ be independent observations in $\mathbb{R}^m$ whose distribution is assumed to be symmetric about $0$. Let $a_1$ be the unit vector with the same direction of $\sum_{i=1}^n X_i$ and let $\{\{a_1\}\}$ be the linear subspace generated by $a_1$. We denote by $X_1^{(1)}, \ldots, X_n^{(1)}$ the orthogonal projection of $X_1, \ldots, X_n$ on $\{\{a_1\}\}^\perp$. Let $a_2$ be the unit vector with the same direction of $\sum_{i=1}^n X_i^{(1)}$. Next, we denote by $X_1^{(2)}, \ldots, X_n^{(2)}$ the orthogonal projection of $X_1, \ldots, X_n$ on $\{\{a_1, a_2\}\}^\perp$. Continuing this procedure, we can define $a_1, \ldots, a_m$ almost surely.

Before transforming the data into the unit cube, we always rotate the data so that $a_i$ and $e_i$ have the same direction for every $i = 1, 2, \ldots, m$, where $e_i$ is the unit vector whose $i$-th component is 1. Then the final data transformed into the unit cube are invariant under any rotation of the original data, since the mean directions are equivariant under rotations of the data. Of course, there may be