A GENERALIZATION OF THE RESULTS OF PILLAI

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Abstract. In a recent article Pillai (1990, Ann. Inst. Statist. Math., 42, 157-161) showed that the distribution $1 - E_\alpha(-x^\alpha), 0 < \alpha \leq 1; 0 \leq x$, where $E_\alpha(x)$ is the Mittag-Leffler function, is infinitely divisible and geometrically infinitely divisible. He also clarified the relation between this distribution and a stable distribution. In the present paper, we generalize his results by using Bernstein functions. In statistics, this generalization is important, because it gives a new characterization of geometrically infinitely divisible distributions with support in $[0, \infty)$.

Key words and phrases: Bernstein function, Laplace-Stieltjes transform, infinite divisibility, geometric infinite divisibility, Lévy process.

1. Introduction and results

Pillai (1990) showed that the distribution $1 - E_\alpha(-x^\alpha), 0 < \alpha \leq 1; 0 \leq x$, where $E_\alpha(x) = \sum_{n=0}^{\infty} x^n / \Gamma(1 + n\alpha)$ is the Mittag-Leffler function, is infinitely divisible and geometrically infinitely divisible (for the definition of geometric infinite divisibility, see below). He also showed that this distribution is equal to the distribution of $Z_\alpha(S(1))$, where $Z_\alpha(t)$ is the stable process with $\mathbb{E} \exp\{-uZ_\alpha(t)\} = \exp\{-tu^\alpha\}, u \geq 0$, and $S(t)$ is the gamma process with the density $x^{t-1}e^{-x}dx/\Gamma(t)$, $x > 0$.

The aim of the present paper is to generalize his results by using Bernstein functions. In statistics, this generalization is important, because it gives a new characterization of geometrically infinitely divisible distributions with support in $[0, \infty)$.

A $C^\infty$-function $f$ from $(0, \infty)$ to $\mathbb{R}$ is said to be a Bernstein function, if $f(x) \geq 0, x > 0$, and $(-1)^p df^p/dx^p \leq 0, x > 0$, for all integers $p \geq 1$ (cf. Def. 9.1 of Berg and Forst (1975)). Thus $df/dx$ becomes a completely monotone function. Such a function $f$ is characterized by

$$f(x) = a + bx + \int_0^\infty (1 - e^{-sx})\mu(ds), \quad x > 0,$$

where $a, b$ are non-negative constants and $\mu(ds)$ is a positive measure on $(0, \infty)$.
such that
\[ \int_0^\infty \frac{s}{1+s} \mu(ds) < \infty \]
(see Theorem 9.8 of Berg and Forst (1975)). In the present paper we assume that
\[ (1.1) \lim_{x \to 0} f(x) = 0, \quad \lim_{x \to \infty} f(x) = \infty. \]
It is easy to see that \( \lim_{x \to 0} f(x) = 0 \) if and only if \( a = 0 \), and that \( \lim_{x \to \infty} f(x) = \infty \) if and only if \( b > 0 \) or \( \mu((0, \infty)) = \infty \). Then, since \( f \) is a non-zero Bernstein function, \( f(x) > 0, x > 0 \), and \( 1/f \) is completely monotone (cf. Exercise 9.9 of Berg and Forst (1975)). Thus, there exists a unique positive measure \( W(dx) \) on \([0, \infty)\) such that
\[ (1.2) \frac{1}{f(x)} = \int_0^\infty e^{-sx} W(ds), \quad x > 0. \]
We denote by \( W^{n*}(dx) \) the \( n \)-times convolution measure of \( W(dx) \). For \( \lambda > 0 \) define the function \( U_\lambda(x) \) on \( \mathbb{R} \) by
\[ (1.3) U_\lambda(x) = \begin{cases} -\sum_{n=1}^\infty (-\lambda)^n W^{n*}([0, x]), & x \geq 0, \\ 0, & x < 0. \end{cases} \]
We shall show that \( U_\lambda(x) \) is a distribution function and it is a generalization of the distribution function \( 1 - E_\alpha(-x^\alpha), 0 < \alpha \leq 1; \ 0 \leq x \). Remark that the function \( f(x) = x^\alpha \) is a Bernstein function. In this case \( W^{n*}([0, x]) = \{1/\Gamma(1+n\alpha)\} x^{n\alpha} \) and \( U_1(x) = 1 - E_\alpha(-x^\alpha) \).

Now, we state the main results of the present paper. All theorems of this section are proved in Section 2. The first theorem shows that \( U_\lambda \) is infinitely divisible.

**THEOREM 1.1.** Let \( f \) be a Bernstein function with (1.1). Then, for every \( \lambda > 0, U_\lambda \) is an infinitely divisible distribution with the Laplace-Stieltjes transform \( \lambda (\lambda + f(\cdot))^{-1} \).

In Theorem 1.2 below, we construct the Lévy process which has the distribution \( U_\lambda \) for \( t = 1 \). This theorem corresponds to Theorem 4.3 of Pillai (1990), which clarifies the relation between the distribution \( 1 - E_\alpha(-x^\alpha) \) and a stable process.

**THEOREM 1.2.** Let \( f \) be a Bernstein function with (1.1). Then, the Lévy process with the distribution function \( U_\lambda \) for \( t = 1 \) is \( Z(S_\lambda(\cdot)) \). Here \( Z(t) \) is the non-negative and non-decreasing Lévy process such that \( E \exp\{-uZ(t)\} = \exp\{-tf(u)\} \) for \( u \geq 0; S_\lambda(t) \) is the gamma process with the probability density \( \lambda^s s^{t-1} e^{-\lambda s} ds/\Gamma(t), s > 0; Z \) and \( S_\lambda \) are independent.

Next, we show that \( U_\lambda, \lambda > 0, \) is geometrically infinitely divisible. A distribution function \( G \) with \( G(0) = 0 \) is said to be geometrically infinitely divisible if for