On a New Boundary Element Solution Scheme for Elastoplasticity*

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Summary: A new boundary element discretization scheme is proposed for three-dimensional elastoplasticity. The governing equation of the problem is transformed into a set of integral equations in the domain and on the boundary. The resulting integral equations are discretized by means of boundary elements as well as volume elements. The system of linear equations thus obtained is solved simultaneously. The formulation is made in terms of increments and also Green's functions available in the literature for elastostatics. Since the domain is also discretized into volume elements in the proposed solution scheme, the incremental solutions at every loading step can be obtained without any iterative procedure.

1 Introduction

The boundary element method is a numerical method of solution in which the governing differential equation of the problem under consideration is transformed into an integral equation on the boundary, and this integral equation is discretized by a finite number of elements located on the boundary. The discretized system includes only the quantities on the boundary, and hence the method can reduce the dimensions of the final set of simultaneous equations to be solved. The boundary element method, therefore, is increasingly attracting the attention of research workers, as the analyzing system based on a "domain-type" method of solution such as the finite difference or the finite element method has become larger and larger. While Brebbia's text book [1] may be recommended as a comprehensive introduction to the boundary element method, the advances and developments of the method can be traced in Mendelson's review [2] and the proceedings [3, 4] of the recent international conferences.

The authors recently proposed a time-space boundary element method for some transient problems, such as the heat conduction [5] and the transient thermal stresses [6]. In the proposed formulation the governing integral equation on the boundary is discretized by a series of time-space boundary elements including higher-order interpolation functions. It was also shown [7] that the proposed method is applicable to the general boundary-value problems expressed by a linear differential operator.

As for the problems of physical nonlinearities due to plasticity and/or creep, the incremental or the rate-type boundary element formulation was presented together with some calculated

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results [8–13]. In these investigations, however, the contributions of inelastic deformation are accounted by a volume integral appearing in the governing integral equation. That is, the effects due to inelastic deformation are regarded as the pseudo body forces. Since these effects are related to the displacement increment in the inner domain of interest, the methods of solution which are now available in the literature have to adopt an iterative procedure to obtain a rigorous solution for every loading step.

In this work, the authors present a new boundary element discretization scheme for elasto-plasticity without any iterative procedure. In the proposed method the inner domain as well as the boundary are discretized into a series of elements. The unknown displacement increments are, therefore, taken out from the volume integrals of the governing integral equations, and they are explicitly accounted in the final set of linear simultaneous equations.

In what follows, we shall first show the governing differential equation of the problem together with the boundary conditions. Then the governing integral equations are derived from a weighted residual statement in which the fundamental solution in elastostatics is used as the weight function. These integral equations are discretized by means of boundary-volume elements, and a final set of simultaneous equations is obtained.

2 Governing Differential Equation

Let us consider the elastoplastic problem of a body which occupies a finite domain \( \Omega \) with the boundary \( \Gamma \) in the three-dimensional space.

Under the assumption of infinitesimal small deformation and strain, the equilibrium equation of the quasi-static elastoplastic problem under consideration can be expressed in the following incremental form

\[
\text{div} \Delta \mathbf{S}(\mathbf{x}) + \Delta \mathbf{b}(\mathbf{x}) = 0 ,
\]

where \( \mathbf{S}(\mathbf{x}) \) and \( \mathbf{b}(\mathbf{x}) \) are the stress tensor and the prescribed body force vector at an arbitrary point \( \mathbf{x} \) of the domain, respectively, while \( \Delta \) denotes an incremental variable. The operator \( \text{div} \) means the divergence with respect to the spatial coordinates \( \mathbf{x} \).

It is assumed that the constitutive relation of the elastoplastic material considered is given by

\[
\Delta \mathbf{S}(\mathbf{x}) = (E + E^p) : \mathbf{\Delta e}(\mathbf{x}) ,
\]

where \( \mathbf{\Delta e}(\mathbf{x}) \) is the strain increment tensor of the point \( \mathbf{x} \). The coefficients \( E \) and \( E^p \) denote the elastic moduli tensor and the plastic moduli tensor, respectively. The latter is obtainable from an assumed yield function and its associated flow rule. The derivation of the plastic moduli tensor is summarized in the Appendix. The colon and the dot placed between the tensorial variables, the latter of which will be encountered later, denote a summation over the repeated two and one index, respectively. They are expressible for a rectangular cartesian coordinate system in the following componental form

\[
[\mathbf{A} : \mathbf{B}]_{ijkl...} = A_{ikl}B_{ijkl...} ,
\]

\[
[\mathbf{A} \cdot \mathbf{B}]_{ijkl...} = A_{ikl}B_{ijkl...} .
\]

The strain increment tensor \( \mathbf{\Delta e}(\mathbf{x}) \) is related to the displacement increment vector \( \mathbf{\Delta u}(\mathbf{x}) \) in the following manner

\[
\mathbf{\Delta e}(\mathbf{x}) = \frac{1}{2} [\text{grad} \mathbf{\Delta u}(\mathbf{x}) + (\text{grad} \mathbf{\Delta u}(\mathbf{x}))^T] ,
\]

where \( \text{grad} \) denotes the gradient with respect to the coordinates \( \mathbf{x} \) and the superfix \( T \) the transpose of a tensor.

Substituting (2) and (3) into (1), we obtain

\[
\text{div} [E : \text{grad} \mathbf{\Delta u}(\mathbf{x})] + \text{div} [E^p : \text{grad} \mathbf{\Delta u}(\mathbf{x})] + \Delta \mathbf{b}(\mathbf{x}) = 0 .
\]

Equation (4) is the governing equation of the problem expressed in terms of the displacement increment.