Thin plate bending problem of dissimilar strips with two bond lines

N. Hasebe, M. Salama

Summary A solution to the thin plate bending problem of partially bonded dissimilar strips with two bond lines is presented. The two strips are symmetrically bonded with respect to the interface which is on the X-axis. The complex stress functions approach together with the rational mapping function technique are used in the analysis. A concentrated bending moment applied at each strip is considered. Distributions of bending and torsional moments, as well as the stress intensity of debonding (SID) at the debonding tips are obtained, and the debonding extension is investigated.

Key words thin plate bending, dissimilar strips, complex stress functions, interface debonding, stress intensity of debonding

1 Introduction
The fracture analysis of composite structure interfaces is certainly important since it provides many design-relevant concepts for the prediction of structural failures. Debonding, which is a phase of failure, has been widely observed in composite structures involving interfaces where the thin plate bending assumptions are justified. This problem in reference to bi-material media is a special case of the composite structure debonding. The strip-shaped bi-material medium is considered here as a test specimen. The problem of thin plate bending of bi-material strips with one bond line has been solved in [1, 2] for two cases of loading, namely, a concentrated bending moment applied at each strip, and a non-uniform change of temperature. However, the two-bond lines problem is more complicated from the mathematical point of view. It has relatively complicated forms of integral equations which arise as a result of the constants of integration representing the resultant force and moment on the bond lines. These constants only appear if the analyzed region is doubly or more connected. Also, since the Plemelj function of a doubly connected area is more complicated, and as some of these integral equations involve the Plemelj function explicitly, the solution is not likely to be done analytically, and a numerical integration technique has to be adopted. The main objective of the present paper is to derive the complex stress functions for the problem of thin plate bending of bi-material strips with two bond lines, using the rational mapping function technique with which arbitrary configurations can be analyzed. The solution of the Riemann-Hilbert problem, like the present case, can only be done with rational mapping functions [3]. This is why the mapping function obtained here has been rationalized in the form expressed in Eq. (1). Bonded dissimilar strips are analyzed for a concentrated bending moment applied at each strip. Distributions of bending and torsional moments along the bonded and unbonded boundaries of the strips are obtained. The stress intensity of debonding (SID), which is used herein to determine the strength of bending and torsional moments that may cause the propagation of debonding at the debonding tips, is derived in a way similar to that of the stress intensity factors. SID is used here to distinguish from the stress intensity factor, since it is calculated on the interface between the two materials. Using SID, the propagation of debonding can be investigated.

2 Analytical method
Figure 1(a) shows two dissimilar strips with different flexural rigidities and material constants, bonded symmetrically with respect to the interface which is on the X-axis. The material at \( Y \geq 0 \) is referred.

Received 12 September 1994; accepted for publication 12 March 1995

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Material 1 rotates about the x-axis as shown in Fig. 1(b), so as to facilitate the analysis scheme which is carried out accordingly. Symbols and subscripts of stress and displacement components are indicated by capital letters in Fig. 1(a), and by small letters in Fig. 1(b). A mapping function by means of which materials 1 and 2 are mapped into the unit circles of the \( t_j \)-planes of Fig. 1(c), \( j = 1, 2 \), respectively, is expressed as follows:

\[
z_j = \frac{w}{\pi} \sqrt{2} \int \frac{dt_j}{(1 - t_j)^{1/2} (1 + t_j)^{1/2} (1 - \tau_j)} = \sum_{k=1}^{N} \frac{E_k}{\zeta_k - t_j} + E_c \equiv \omega(t_j),
\]

where \( E_k, \zeta_k, (k = 1, \ldots, N), N = 60 \) and \( E_c \) are complex constants, and \( w \) is the width of the strips. The procedure of deriving the rational mapping function is not given in this paper for the sake of brevity, and it has been thoroughly explained in [4].

On the interface of Fig. 1(a), \( Z_1 = Z_2 \), which corresponds to \( t_1 = t_2 \) on the unit circles. Thus, the boundary conditions on the interface are expressed in terms of a single variable \( \sigma \), which satisfies \( \sigma = 1/\bar{\sigma} \) on the unit circle. The bonded boundaries are denoted by \( M_g, (g = 1, 2) \), while the unbonded boundaries are denoted by \( L_g, (g = 1, 2) \), Fig. 1(b).

Bending moments, torsional moments, deflections, bending forces (effective shear forces) and rotations in Fig. 1(a) and 1(b) are related as follows:

\[
M_{y_1} = -m_{y_1}, \quad M_{y_2} = m_{y_2}, \quad M_{x'_1} = -m_{z_1}, \quad M_{x'_2} = m_{z_2}, \quad M_{x_1} = m_{x_1}, \quad M_{x_2} = m_{x_2}, \quad M_{x_1} = m_{x_1}, \quad M_{x_2} = m_{x_2},
\]

\[
W_1 = -w_1, \quad W_2 = w_2,
\]

\[
\int P_y \, ds = \int p_y \, ds, \quad \int P_z \, ds = \int p_z \, ds,
\]

\[
\frac{\partial W_1}{\partial x_1} = -\frac{\partial w_1}{\partial x_1}, \quad \frac{\partial W_1}{\partial x_2} = -\frac{\partial w_1}{\partial x_2}, \quad \frac{\partial W_1}{\partial y_1} = -\frac{\partial w_1}{\partial y_1}, \quad \frac{\partial W_1}{\partial y_2} = -\frac{\partial w_1}{\partial y_2}.
\]

(2)