A unified treatment of the elastic elliptical inclusion under antiplane shear

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Summary A generalized and unified treatment is presented for the antiplane problem of an elastic elliptical inclusion undergoing uniform eigenstrains and subjected to arbitrary loading in the surrounding matrix. The general solution to the problem is obtained through the use of conformal mapping technique and Laurent series expansion of the associated complex potentials. The resulting elastic fields are derived explicitly in both transformed and physical planes for the inclusion and the surrounding matrix. These relations are universal in the sense of being independent of any particular loading as well as the geometry of the matrix. The complete field solutions are provided for an elliptical inclusion under uniform loading at infinity, and for a screw dislocation interacting with the elastic elliptical inclusion.

Keywords Antiplane shear, eigenstrains, inclusions, inhomogeneities, complex potentials

1 Introduction
The inclusion problem is known to play a key role in the micromechanical analysis of the mechanical behavior of materials [1]. Donnell [2] appears to be the first to address the problem of a homogeneous elliptical inclusion which possesses the same elastic properties as the surrounding matrix and undergoes uniform stress-free transformation strains (eigenstrains). On the other hand, Hardiman [3] considered the case of a two dimensional elliptical inhomogeneity having elastic properties different from those of the matrix. Eshelby [4, 5] later provided a systematic investigation of the corresponding three-dimensional problems. He introduced the point-force concept for solving 3D inclusion problems and also developed an ingenious method known as the equivalent inclusion method for dealing with 3D inhomogeneity problems. Most subsequent studies on inclusion and inhomogeneity problems have followed the Eshelby's approach and many solutions are listed in Mura's book [1]. Eshelby's solutions generally involve a set of analytically intractable integrals and for plane and antiplane problems, the complex variable method in both isotropic and anisotropic elasticity is often adopted to facilitate the determination of explicit solutions [6–12]. The equivalent inclusion method is most effective for the particular case of an inhomogeneity under uniform remote loading. For more general loading conditions, other methods have to be developed. Some circle theorems have been revealed for the treatment of circular inhomogeneities under arbitrary loading conditions in an infinite matrix [13–16]. Universal relations have also been recently derived for an elliptical inhomogeneity under arbitrary antiplane and plane loading in a finite or infinite matrix [17, 18].

In the present study, we provide a unified treatment of the elastic elliptical inclusion which undergoes uniform eigenstrains and is subjected to arbitrary antiplane loading in the surrounding matrix. The analysis is based upon the use of conformal mapping and the Laurent series expansion method. A stress-free displacement potential is introduced to facilitate the complex variable formulation. The general forms of the stress functions in the inclusion and the sur-
rounding matrix are derived explicitly in both transformed and physical planes. These expressions are universal in the sense of being applicable to arbitrary loading conditions and being valid for both finite and infinite media. Furthermore, both inclusion and inhomogeneity problems can be treated within the same framework using the proposed approach. Complete field solutions are provided for an inhomogeneous inclusion under uniform remote loading as well as for a screw dislocation interacting with the elliptical inclusion. The results contain many previous known solutions as special cases.

2 General formulation

For antiplane shear deformation, the only displacement $w$ along the $z$-axis satisfies Laplace’s equation

$$\Delta^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

under conditions of equilibrium. The shear stress components $\tau_{xz}$, $\tau_{yz}$ and strain components $\varepsilon_{xx}$, $\varepsilon_{yy}$ are related by Hooke’s law as

$$\tau_{xz} = 2\mu \varepsilon_{xx} = \mu \frac{\partial w}{\partial x}, \quad \tau_{yz} = 2\mu \varepsilon_{yy} = \mu \frac{\partial w}{\partial y},$$

where $\mu$ is the shear modulus of the material. It is known that the displacement $w$, shear stresses $\tau_{xz}$ and $\tau_{yz}$ and the resultant force $T$ along any arc $AB$ in the material can be expressed in terms of a single analytical function of the complex variable $z = x + iy$ as [17]

$$w = \frac{1}{2\mu} [\Phi(z) + \overline{\Phi}(z)],$$

$$\tau_{xz} - i\tau_{yz} = \Phi'(z),$$

$$T = \int_{A}^{B} (\tau_{xz} \, dy - \tau_{yz} \, dx) = \frac{i}{2} [\Phi(z) - \Phi(z)]_{A}^{B},$$

where the overbar represents the complex conjugate, the prime denotes the derivative with respect to the argument and $[\ ]_{A}^{B}$ denotes the change in the bracketed function in moving from point $A$ to point $B$ along the arc. In using Eq. (5), the direction of increasing arc-coordinate must be chosen so that the material is on the left when moving along this direction.

Consider now the antiplane problem of an isotropic elastic matrix material containing an elliptical inclusion which undergoes the prescribed uniform stress-free transformation strains (or eigenstrains) $\varepsilon_{xx}^{T}$ and $\varepsilon_{yy}^{T}$ and possesses different elastic properties from the matrix, Fig. 1. The elliptical interface is assumed to be perfectly bonded along the common boundary $L$. The regions occupied by the elastic matrix and the inclusion will be referred to as regions 1 and 2, respectively, and the quantities associated with these regions will be denoted by the corresponding subscripts.

In the absence of the surrounding matrix, the uniform eigenstrains $\varepsilon_{xx}^{T}$ and $\varepsilon_{yy}^{T}$ would lead to the following prescribed displacement $w^{T}$ through eq. (2)

$$w^{T} = 2x\varepsilon_{xx}^{T} + 2y\varepsilon_{yy}^{T} = xe^{T} + ye^{T}, \quad \text{with} \quad e^{T} = \varepsilon_{xx}^{T} - i\varepsilon_{yy}^{T}. \quad (6)$$

Fig. 1. An elliptical inclusion in physical and transformed planes.