Regularized computation of interior displacements and stresses by BEM

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Summary  The computation of interior displacements and stresses with the boundary element method (BEM) often requires the evaluation of nearly singular integrals. These integrals arise from the singular behaviour of the kernel functions in the Somigliana identity and the Somigliana stress identity. Treating them numerically in a standard way leads to inaccuracy near the boundary. This effect is always present in the calculation of field variables near the boundary and is called 'boundary layer effect'. In this paper regularization procedures are proposed which consist of an indirect evaluation of singular integrals and a special coordinate transformation. The proposed procedures eliminate the boundary layer effect for both, the calculation of displacements and stresses. In a numerical example of elastostatics the developed strategies are shown to work. Due to the generality of the proposed procedures they can be extended to any standard boundary element formulation for problems with bounded domains.

Key words  Boundary element method, regularization, stress analysis

1 Introduction
If the boundary data of a given problem have been calculated by BEM this problem is entirely solved because the domain variables are governed by the boundary variables. In a postprocessing step all field variables in the interior can be determined from the boundary solution. However, if the interior load point is located close to the boundary, the calculation of field variables fails. This is due to the lack of standard numerical integration schemes to integrate nonregular functions. Such nearly singular functions arise in the computation of displacements near the boundary with the Somigliana identity and in the computation of stresses with the Somigliana stress identity. Due to the higher order of singularity of the Somigliana stress identity the problems are even more severe.

In order to circumvent these problems, integral statements of less singularity order have been developed in the last decade beginning with an early paper by Sládek and Sládek [14]. They reduce the order of singularity in the stress equation by using Stokes' theorem. On the other hand Ghosh et al. [8] use integration by parts to weaken the strongly singular kernel in Somigliana's identity for two-dimensional elastostatics. Thus, the corresponding stress equation is only strongly singular. Another approach has been presented by Okada et al. [13]. They introduce a different test function in the weighted residual statement and also end up with a less singular integral statement. All these formulations have in common, that they introduce displacement derivatives instead of displacements on the boundary. This necessitates shape function differentiation or reformulation of the original problem. Therefore, a regularized formulation that is based on the original direct BEM approach seems to be more meaningful. This is obtained by considering relative quantities which can also be interpreted as a subtraction technique. For the first subtracted term it is equivalent to the rigid body motion argument, although it is not necessary to make use of this interpretation [4]. Such formulations were introduced by Kisu and Kiwahara [11] for the calculation of potentials near the boundary and further developed by Crotty Sisson [3] also for the calculation of stresses.

In this paper a procedure is proposed, which, for the calculation of displacements consists of a rigid body motion technique. An efficient indicator is proposed, which decides if the interior load point lies in the boundary layer. The stress equation is partially regularized such that the boundary layer indicator
can also be used for the stress calculation. Note, that due to the different order of singularity in the standard formulation, the boundary layer in the displacement equation and in the stress calculation have different thickness. For full regularization of the Somigliana stress identity a new self-adaptive coordinate transformation is introduced which provides very accurate results near the boundary.

2 Integral formulations

The basic integral equation in elastostatics is Somigliana’s identity [2]

\[ u_r(\xi) = \int_{\Gamma_s} \hat{u}_q(x, \xi) p_j(x) \, d\Gamma_s - \int_{\Gamma_s} \hat{p}_q(x, \xi) u_j(x) \, d\Gamma_s. \]  

In (1) \( x \) are field points, \( \xi \) is the load point, \( u_r(x) \) and \( p_j(x) \) denote displacements and tractions on the boundary and \( \hat{u}_q \) and \( \hat{p}_q \) are Kelvin’s fundamental solutions of displacements and tractions, respectively, given by

\[ \hat{u}_q = \frac{1}{16 \pi \mu(1-v) r} ((3-4v) \delta_{qj} + r_j r_q), \]  

\[ \hat{p}_q = \frac{-1}{8 \pi (1-v) r^2} \left[ ((1-2v) \delta_{qj} + 3 r_j r_q \frac{\partial}{\partial n} + (1-2v)(r_j n_i - r_i n_j) \right], \]  

where \( r \) denotes the Euclidian distance between load point \( \xi \) and field point \( x \), \( r = \sqrt{(x_i - \xi_i)(x_i - \xi_i)} \).

The corresponding integral statement for the calculation of stresses at interior points is obtained by differentiating (1) with respect to the coordinates of \( \xi \) to produce the strain tensor and then substituting the result into the constitutive law. The final expression is the Somigliana stress identity

\[ \sigma_{ij}(\xi) = \int_{\Gamma_s} \hat{u}_{ik}(x, \xi) p_k(x) \, d\Gamma_s - \int_{\Gamma_s} \hat{p}_{ik}(x, \xi) u_k(x) \, d\Gamma_s \]  

with the kernel functions

\[ \hat{u}_{ik} = \frac{1}{8 \pi (1-v) r^3} \left[ ((1-2v)(\delta_{ik} r_j + \delta_{jk} r_i - \delta_{ij} r_k) + 3 r_j r_i r_k \right], \]  

\[ \hat{p}_{ik} = \frac{\mu}{4 \pi (1-v) r^3} \left[ 3 \frac{\partial}{\partial n} ((1-2v) \delta_{ik} r_k + \nu (\delta_{ik} r_j + \delta_{jk} r_i) - 5 r_j r_i r_k) \right. \]  

\[ + 3 (1-2v) r_j r_i n_k + 3 \nu (r_j n_i + r_i n_j) r_k \]  

\[ - (1-4v) \delta_{ik} n_k + (1-2v)(\delta_{ik} n_i + \delta_{ik} n_j) \right] \]  

as given e.g. in [2].

For \( r \to 0 \) the fundamental tensor \( \hat{u}_q \) behaves weakly singular like \( r^{-1} \), the kernels \( \hat{p}_q \) and \( \hat{u}_{ik} \) are strongly singular like \( r^{-3} \), and \( \hat{p}_{ik} \) is the hypersingular kernel, which implies a \( r^{-3} \) singularity. As has been pointed out by many authors (see e.g. [1], [3], [4], [7], [15]) the singular behaviour of the kernel functions leads to inaccurate results for interior point computations near the boundary, the so called ‘boundary layer effect’ shows up. This requires special treatment of the integral equations (1) and (4).

3 Regularization of the Somigliana identity

In order to demonstrate the efficiency of the employed regularization procedure the example of a longitudinally loaded cube (Fig. 1) is chosen. This cube is discretized with 6 linear boundary elements and the boundary solution of this problem is supposed to be computed in a preceding step. In what follows internal points located on the dashed line in Fig. 1 will be considered.