The Concomitants of Spinors of Type [3/2, 1/2] in Space-Time

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Abstract

Explicit forms for the concomitants which are bilinear in two spinors of type [3/2, 1/2] and the concomitants which are quadratic in a single spinor of type [3/2, 1/2] are obtained. The dual-tensors, where they exist, are also given.

The concomitants of higher-order spinors can be obtained in an exactly similar manner.

1. Introduction

The definition of spinors is dependent upon the frame of anticommuting matrices. In four-dimensional space-time, where the metric is taken to have signature +2, the anticommuting set of matrices $X_i$ is taken as that used by Littlewood (1972), namely

$$X_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad X_0 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

where

$$X_0^2 = -I, \quad X_i^2 = I$$

$$X_0 = -X_0, \quad X_i = X_i. \quad (i = 1, 2, 3).$$

Defining the metric tensor as

$$g_{ij} = g^{ij} = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j; i, j = 1, 2, 3), \\ -1 & (i = j = 0), \end{cases}$$
it follows that
\[ X^0 = -X_0, \quad X^i = X_i, \quad (i = 1, 2, 3), \]
and hence
\[ (x^i x^i)^2 = (x^i x_i)^2 = g_{ij} x^i x^j = g^{ij} x_i x_j, \]
which will be taken as the metric of space-time.
Corresponding to any Lorentz transformation, \( L \) say, as given by
\[ x'_i = a_i^j x_j, \quad (i = 1, 2, 3, 0), \]
there is a matrix \( U \), the basic spin matrix, which is unique apart from sign such that
\[ a_i^j x_j = U^{-1} x_i U, \]
where \([a_i^j]\) is a Lorentz matrix whose elements thus satisfy the 'orthogonal' relations
\[ a^i_k a^j_l = g_{kl}, \quad a^i_l a^j_k = g^{ij}. \quad (1.2) \]
A four-rowed real vector which, under \( L \), is transformed by \( U \), is called a basic spinor. By considering the direct product of a simple tensor of type \( \{n\} \) and a basic spinor, on removal of the contractions present, an irreducible symmetric spinor of type \([n + 1/2, 1/2]\) is obtained, \( n \) being a positive integer. Explicit forms for such spinors have been given elsewhere (Dodds, 1972). If \( V_1, \ldots, v_n \) is an irreducible symmetric spinor of type \([n + 1/2, 1/2]\), it consists of \((n + 3)!/n!3! \) real four-vectors of which just \((n + 2)!/n!2! \) are independent because of the zero contractions \( X^i V_1, \ldots, v_n = 0 = g^{ij} V_1, \ldots, v_n \).

The concomitants which are bilinear in two spinors of type \([n + 1/2, 1/2]\) and the concomitants which are quadratic in a single spinor of type \([n + 1/2, 1/2]\), are of types given by the expansions of the products \([n + 1/2, 1/2][n + 1/2, 1/2] \) and \([n + 1/2, 1/2] \otimes \{2\} \) respectively. In the case of basic spinors, i.e. when \( n = 0 \), it is well known that the concomitants which are bilinear in two basic spinors are five in number, consisting of an invariant, a pseudo-invariant, a four-vector, a pseudo four-vector and a six-vector. When the basic spinors are made equal to give the concomitants which are quadratic in a single basic spinor, just the four-vector and the six-vector survive. This paper is concerned with the analysis of the higher-order case when \( n = 1 \), i.e. when the spinor or spinors are of type \([3/2, 1/2]\).

In order to illustrate later results, a particular reference frame is used. Suppose that \( W \) and \( Z \) are two basic spinors and that \( \xi_1 \) and \( \eta_1 \) are two tensors of type \( \{1\} \). Putting \( W_1 = \xi_1 W \) and \( Z_1 = \eta_1 Z \), then \( V_1 \) and \( Y_1 \), where \( X^i V_1 = 0 = X^i Y_1 \), are two irreducible spinors of type \([3/2, 1/2]\) where (Dodds, 1972)
\[ V_1 = W_1 - X_1 (X^j W_j)/4, \quad Y_1 = Z_1 - X_1 (X^j Z_j)/4. \quad (1.3) \]
Following Littlewood (1969, 1972), the particular reference frame is chosen in which one of the basic spinors, \( Z \) say, is in canonical form. Hence, in this reference frame, the basic spinors are given by
\[ \vec{W} = [\alpha, \beta, \gamma, \delta], \quad \vec{Z} = [\epsilon, 0, 0, 0], \]