Quantization in Generalized Coordinates
III—Lagrangian Formulation

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Abstract

It is shown that if one incorporates the generalized coordinate quantum velocities \( \dot{q}^i \) as given by \( \dot{q}^i = i[H,q^i] \) \((\hbar = 1)\) into the generalized classical Lagrangian for a free particle (the total energy), \( L = \frac{1}{2} g_{ik} \dot{q}^i \dot{q}^k \), one does not obtain (no matter what ordering of the operators \( \dot{q}^i, \dot{q}^k \), and \( g_{ik} \) we choose) the correct quantum Lagrangian operator which is a transformation from \(-i\nabla^2\) to generalized coordinates (Gruber, 1971, 1972). \( \dot{q}^i \) as given by \( \dot{q}^i = i[H,q^i] \) turns out to be the Hermitian part of a more generalized operator which we call the total generalized velocity operator similar to the notation in our previous articles (Gruber, 1971, 1972). This total velocity operator really determines the fundamental structure governing our system in the Lagrangian formulation. We show that it is through the total velocity operator that we make the transition from classical to quantum mechanics and through our procedure we arrive at the correct quantum Lagrangian operator.

1. Introduction

In two previous articles, I (Gruber, 1971) and II (Gruber, 1972), I have shown a prescription for the transition of classical quantities to their corresponding quantum operators in generalized coordinates. These prescriptions deal with, representing quantum mechanically, operators in generalized coordinates corresponding to classical functions of generalized momenta and coordinates, such as the Hamiltonian of the system. Difficulty arises when one tries to represent in generalized coordinates quantum mechanical operators corresponding to functions of generalized velocities and coordinates. For example, consider the following: The total energy of a free particle (the Lagrangian) expressed in generalized coordinates is classically given by Brillouin (1949) (throughout this article, repeated indices denote Einstein summation)

\[
L = \frac{1}{2} g_{ik} \dot{q}^i \dot{q}^k
\]
where \( g_{ik} \) are functions of the generalized coordinates and \( \dot{q}^i \) is the generalized velocity. Now if one substitutes the operator \( \dot{q}^i \) given by the familiar equation

\[
\dot{q}^i = i[H, q^i] = i(Hq^i - q^i H)
\]  

(1.2)

into the Lagrangian of equation (1.1), no matter what the ordering we choose for \( \dot{q}^i, q^k \), and \( g_{ik} \) (that is, if \( L = \frac{1}{2} \dot{q}^i g_{ik} \dot{q}^k \) or \( L = \frac{1}{2} q^i \dot{q}^k g_{ik} \), etc.) we will not arrive at the correct quantum Lagrangian operator which is given as (Gruber, 1972)

\[
L = \frac{1}{2} \dot{q}^i \left( g^{ik} \frac{\partial}{\partial q^k} \right)
\]  

(1.3)

where \( g \) is the Jacobian \( \left[ \frac{\partial x'^i}{\partial q^k} \right] \) (Sokolnikoff, 1951) of the transformation from Cartesian to generalized coordinates and \( g^{ik} \) is the contravariant metric tensor (Gruber, 1972).

In the following sections we will proceed to find what the fundamental velocity operator is and how to incorporate it into the classical generalized Lagrangian to get the correct quantum-mechanical Lagrangian operator. We will also show just what the real significance of the operator \( \dot{q}^i \), given as \( \dot{q}^i = i[H, q^i] \), is.

2. Representation of Generalized Velocities in Quantum Theory

Consider the classical Lagrangian expressed in generalized coordinates:

\[
L = \frac{1}{2} g_{ik} \dot{q}^i \dot{q}^k
\]  

(1.1)

The generalized classical momentum \( p_i \) is given as

\[
p_i = \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^k g_{ik}
\]  

(2.1)

Multiplying both sides of equation (2.1) by \( g^{ij} \), we obtain (classically)

\[
\dot{q}^j = g^{ij} p_i
\]  

(2.2)

This is because

\[
g_{ik} g^{kj} = \delta^j_i, \quad j \neq i
\]

Now, quantum-mechanically, we postulate (Gruber, 1971, 1972) that \( p_i \rightarrow -i\partial/\partial q^i \) \((h = 1)\). Thus we note that quantum-mechanically, \( \dot{q}^j \) can be written as either

\[
\dot{q}^j = g^{ij} p_i
\]  

(2.3)

or as

\[
\dot{q}^j = p_i g^{ij}
\]  

(2.4)

since \( p_i \) and \( g^{ij} \) do not commute.

† Here, \( H \) is the Hamiltonian given by \( H = -\frac{i}{2} V^2 + \frac{1}{2} p^i g^{ij} p_j \) (Gruber, 1971, 1972).