Helmert’s projection of a ground point onto the rotational reference ellipsoid in topocentric cartesian coordinates

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Summary. In order to derive the ellipsoidal height of a point $P_t$ on the physical surface of the earth, and the direction of ellipsoidal normal through $P_t$, we present here an iterative procedure rapidly convergent to compute, in a topocentric Cartesian system, the coordinates of Helmert’s projection of the ground point $P_t$ onto the reference ellipsoid of revolution. We derive as well the cofactor matrix of total vector of the topocentric coordinates of the above ground point and of its Helmert’s projection so that to compute the variance of ellipsoidal height.

Introduction

Usually the projection of a ground point onto the reference ellipsoid, along the ellipsoidal normal, is performed by inverting the relationships between the geodetic coordinates $(\phi, \lambda, h)$ and the corresponding geocentric Cartesian coordinates $X, Y, Z$. For this inversion there are many procedures:

- Iterative processes are shown in Geodesy text-books (Heiskanen and Moritz, 1967, Vanicek and Krakiwsky, 1986, Torge, 1991) and papers (Ferrara and Giannoni, 1985).

- Bowring (1985) proposes a closed approximate formula which is particularly improved for the accuracy of an outer space-situation.

- Exact closed solutions have been derived which lead to the problem of solving an algebraic equation of fourth order (Ungueudoli, 1974, Vanicek and Krakiwsky, 1986).

- Borkowski (1989) performs an analysis of several algorithms and proposes two accurate closed solutions of which one is approximative and the other is exact.

- Grafarend and Lohe (1991) have recently proposed a procedure based on the theory of degenerate conics, which has to solve a stable equation of third order and a system of linear equations. In their procedure Helmert’s projection takes only geocentric Cartesian coordinates.

Obviously, if the topocentric Cartesian coordinates only of ground point are known, in order to adopt the above mentioned procedures we need at first to perform the rototranslation of topocentric system to geocentric system.

For the spatial determination of points on the physical surface of the earth with respect to the rotational ellipsoid, it may be introduced the topocentric Cartesian coordinates which are useful above all in control surveys involving areas of limited extent. Furthermore, in tridimensional adjustment of spatial networks, the observation equations may be written and linearized taking directly as unknown parameters the topocentric Cartesian coordinates of points. Hence the need to solve the problem of Helmert’s projection of a ground point directly using topocentric Cartesian coordinates, to compute as the components of unit normal vector as the ellipsoidal heights.

Therefore in this paper we obtain first the equation of rotational ellipsoid in a topocentric system. Soon after it is shown an iterative algorithm, rapidly convergent, for Helmert’s projection directly in topocentric Cartesian coordinates. Finally in order to compute the variance of ellipsoidal height we derive the cofactor matrix of total random vector of the ground point coordinates as well as those of Helmert’s projection.

1 Equation of rotational ellipsoid in Topocentric system

Given a rotational ellipsoid $\sigma$, let $OXYZ$ be the geocentric Cartesian right-handed system whose origin $O$ is at the ellipsoid center and $Z$-axis coincides with rotational axis; the $X$- and $Y$-axes lie in the plane of the Equator, with $X$-axis in meridian plane of reference for ellipsoidal longitudes. Let $P_0$ be the topocentric Cartesian right-handed system whose origin is the known point $P_0 = (\phi_0, \lambda_0) = (X_0, Y_0, Z_0)$ on the ellipsoid and whose $z$-axis coincides with the ellipsoidal normal at $P_0$, the $x$- and $y$-axes span the plane tangent to ellipsoid at $P_0$ and with $y$-axes rotated of clockwise azimuth $\theta$ with
The coefficients $c_i$ (i = 1,...,7) of equation (4) are shown in Appendix B. Solving (4) you get the explicit equation:

$$z = z_e(x,y) = \frac{- (c_4 + c_6 x + c_7 y) \pm \sqrt{(c_4 + c_6 x + c_7 y)^2 - 4c_3 (c_1 x^2 + c_2 y^2 + c_5 xy)}}{2c_3} \quad (5)$$

where the sign $+$ must be assumed in enough limited area about the origin $P_o$ of topocentric system.

2. Topocentric Cartesian coordinates of Helmert's projection of a ground point onto the ellipsoid

Helmert's projection $P_e$ is obtained by projecting the surface point $P_t$ along the ellipsoidal normal.

![Fig.2. Helmert's projection of the point $P_t$ onto the ellipsoid](image)

Given the topocentric coordinates $(x_t, y_t, z_t)$ of $P_t$, the topocentric Cartesian coordinates $(x_e, y_e, z_e)$ of $P_e$ are real solution of the not linear system:

$$x = x_t + t \cdot F_x$$
$$y = y_t + t \cdot F_y$$
$$z = z_t + t \cdot F_z \quad (6)$$

$$F_x(x - x_e) + F_y(y - y_e) + F_z(z - z_e) = 0 \quad (7)$$

$$F(x, y, z) = c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 x + c_5 y + c_6 xz + c_7 yz = 0 \quad (8)$$

where (6) are the equations of the straight line passing through $P_t$ and orthogonal to ellipsoid in $P_e$, (7) is the equation of the tangent plane to ellipsoid in $P_e$, and (8) is the equation of the ellipsoid, and where

$$F_x = \frac{\partial F}{\partial x} \bigg|_{P_e} = 2c_1 x_e + c_5 y_e + c_6 z_e$$