PLANE WAVES IN THE GENERAL THEORY
OF RELATIVITY

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The Newman–Penrose method is used to study the class of gravitational fields in a vacuum which permit a normal congruence of isotropic geodesics. The energy–momentum tensor is used in tetrad form to prove that if the nondegenerate metric of these fields depends only on a single isotropic coordinate, the solutions will describe plane gravitational waves.

The Newman–Penrose method [2] is a powerful instrument for studying gravitational fields; it has been used to obtain many new exact and approximate solutions of the gravitation equations [4, 5]. In addition, it significantly simplifies the problem of establishing a relationship between the algebraic properties of the gravitational fields and their physical characteristics. This method is used below to study a particular class of gravitational fields in a vacuum which permit a normal congruence of isotropic geodesics; all the metric coefficients are functions of only one isotropic coordinate.

The field equations discussed in [2], which are equivalent to the Einstein equations, are a set of differential equations for a system of variables including the Ricci rotation coefficients, the physical components of the Weyl tensor, and the components of the metric tensor. These equations simplify considerably with a suitable choice of quasiorthogonal tetrad and coordinate system [3].

We consider the case in which the metric depends only on the one isotropic coordinate (the coordinate u in [2]) and in which the congruence of isotropic geodesics of the space is normal. Under these conditions the Newman–Penrose field equations are [2, 4]:

radical equations,

\begin{align}
 a) \rho'^2 + \varphi'^2 &= 0, \\
 b) \rho' + \varphi' &= 0, \\
 c) \varphi' &= 0, \\
 d) \rho' - \varphi' &= 0, \\
 e) \rho' + \varphi' &= 0, \\
 f) \rho' + \varphi' &= 0, \\
 g) \rho' + \varphi' &= 0, \\
 h) \rho' + \varphi' &= 0,
\end{align}

nonlinear equations,

\begin{align}
 a) \rho' &= \lambda', \\
 b) \rho' &= \lambda', \\
 c) \rho' &= \lambda', \\
 d) \rho' &= \lambda', \\
 e) \rho' &= \lambda', \\
 f) \rho' &= \lambda', \\
 g) \rho' &= \lambda', \\
 h) \rho' &= \lambda'.
\end{align}
Bianci identities,

\( a) \ \Psi_0 = (4\gamma - \nu) \Psi_0 - (4\gamma + 2\beta) \Psi_2 + 3\omega \Psi_2, \)
\( b) \ \Psi_1 = \nu \Psi_0 + 2 \gamma \Psi_1 - 3\omega \Psi_2 + 2\psi \Psi_3, \)
\( c) \ \Psi_2 = 2\nu \Psi_1 - 3\nu \Psi_2 - 2\psi \Psi_3 + \omega \Psi_4, \)
\( d) \ \Psi_3 = 3\nu \Psi_2 - 2 \gamma \Psi_3 + 2 \beta \Psi_4. \)

Here the dots denote differentiation with respect to the coordinate \( u \), while the primes denote complex conjugates. The \( \Psi_0 \ldots \Psi_4 \) are the projections of the Weyl tensor (in our case the Riemann tensor) onto the quasiorthogonal tetrad. The functions \( U, \omega, \xi^i, \chi^i \) \((i = 3, 4)\) are related to the metric by

\( g_{11} = g_{22} = 0, \ g_{12} = 2(U - \omega u), \)
\( g_{33} = (\gamma - \nu) \Psi_1 - 2\psi \Psi_3 + \omega \Psi_4, \)
\( g_{44} = - (\gamma \chi^i + \nu \xi^i). \)

The other functions \( \rho, \sigma, \tau \) are components of the Ricci rotation coefficients in the given quasiorthogonal tetrad system; in our case, we have

\( \rho = \bar{\rho}, \ \sigma = \bar{\sigma} + \beta. \)

We turn now to a study of system (1)-(3). Combining (1. i) with a conjugate of (1. h), we find

\( \tau_0 + \tau_2 + \Psi_1 = 0. \)

It follows from (6) and (1. g) that

\( \tilde{\tau}(\rho - \sigma) = 0. \)

If \( \rho = \sigma \) (1. e) yields \( \rho = \sigma = 0 \) and (1. b) yields \( \tau = 0. \) On the basis (3), we therefore set \( \tau = 0. \)

We now consider the case in which \( \alpha = -\beta \neq 0; \) it follows from (1. g, m, j, n) that \( \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0. \)

From (1. q) we have \( \rho \Psi_1 = 0. \) For the case of a nonplanar solution (\( \Psi_4 \neq 0 \)) \( \rho = 0. \) Then it follows from (1. e) that \( \sigma = 0 \) and from (2. g) that \( \alpha = \beta = 0. \)

In studying Eqs. (1)-(3) we can therefore start from the assumption that \( \tau = \alpha = \beta = 0. \)

Then it follows from Eqs. (1. g, j, m, o, p) that

\( \Psi_4 = \Psi_3 = \Psi_0 = 0, \ \gamma \Psi_5 = 0, \ \rho \Psi_4 = 0. \)

If \( \Psi \neq 0, \) we find from the remaining equations that

\( \Psi_4 = 0, \ \rho \xi^i + \sigma \chi^i = 0, \ \rho \mu + \sigma = 0. \)

Multiplying the second relation by \( \rho \) and using the third one, we find

\( \rho \xi^i - \sigma \chi^i = 0. \)

From (9) and (10), we have \( \chi^i = 0; \) in this case, however, the metric of the space becomes degenerate, since the determinant of the metric tensor (4) vanishes.

We are thus left with only one possibility:

\( \Psi_4 \neq 0, \ \gamma = -\tilde{\gamma}, \ \mu = \tilde{\mu}, \ \nu = -\mu^2 - \lambda \tilde{\mu}, \)
\( \xi^i = (2\gamma - \mu) \xi^i - \lambda \tilde{\mu}, \ \omega = (2\gamma - \mu) \omega - \lambda \tilde{\mu} \chi^i, \)
\( \chi^i = -2(2\gamma + \mu) \gamma - \Psi_4. \)

The other coefficients vanish.