The Exponential Distribution in Small Angle X-Ray Scattering. Theory and Practice*

A. Jánosi

Institute of Physical Chemistry, University of Graz, A-8010 Graz, Austria

Summary. From all the theoretical small-angle X-ray scattering (SAXS) curves of compact (non-particulate) systems elaborated systematically by Porod [2], we give a theoretical analysis of only one scattering curve, the corresponding correlation function of which is an exponential distribution. To obtain a correct as well as an easier determination of the zero-intensity $I_0$ and the correlation length $l_c$ than with the procedure usual up to now (analysis of the plot $I(s)^{-1/n}$ vs. $s^2$ with $n = 2$ or 3/2) the classical SAXS-parameters of particle scattering will be involved in the evaluation. In this way the results get also a more useful conception for a practical application.

Keywords. Small-angle X-ray scattering (SAXS); Exponential distribution.

Introduction

Description of the Exponential Distribution

In the zero-order Poisson distribution the random variable $r$ is said to have the standard exponential distribution if its probability density function at $r$, in conventionally abbreviated form, is

$$\gamma(r) = \begin{cases} 0 & \text{for } r < 0 \\ a \exp(-ar) & \text{for } r \geq 0 \end{cases}$$

in which $a$ is an adjustable, positive and real number, called the parameter of the distribution. This distribution is referred to either as the negative exponential or

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simply as the exponential. In the following we use the second version. The expectation value of the distribution is $E(r) = 1/a$ and the variance $V(r) = 1/a^2$.

The exponential distribution is generally well-known to describe the radio-active disintegration or among others the appearing of defects in matter. In his theoretical publication [2] Porod systematically studies the small-angle X-ray scattering (SAXS) curves of various compact (or non-particulate) systems, and shows that in some cases the self-convolution of the electron density distribution in the system, the so-called characteristic function or correlation function corresponds to an exponential distribution as well, e.g. gel-structure with increasing concentration. Earlier Debye and Bueche found the same by the light scattering study of Lucite and two glass samples [1]. Utilizing the exponential distribution as a correlation function in SAXS (or in small-angle scattering in general), the random variable $r$ signifies the distance, measured from an arbitrary point in the matter. The parameter of the distribution, $a$, is now the reciprocal value of a mean distance. This distance is defined [3] as the half of the integral breadth $l_c$ of the correlation function ($l_c/2 = 1/a$). $l_c$ is the so-called coherence- or correlation length defined by Porod [4]. It is known that the reduced* chord length or intersection length $l_r$, defined also by Porod [5], can be obtained by differentiating the correlation function at $r \to 0$. In our case (normalized exponential function) the differentiation always gives [3]

$$\left(\gamma(r)/\gamma(0)\right)' = -a = -1/l_r.$$ 

Therefore, we obtain for the exponential distribution and \textit{only for this distribution}, an important relation between its correlation length $l_c$ and its reduced chord length $l_r$

$$2l_r = l_c$$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{1 the correlation function with exponential distribution $\gamma_0(r) = \exp(2r/l_c), l_c = 10$ nm; 2 the correlation function of a sphere, $\gamma_0(r) = 1 - 3x/2 + x^3/2, x = r/D$, with the diameter $D = 13.3$ nm, corresponding to $l_c = 10$ nm, and 3 Gaussian function with $2\sigma = l_c = 10$ nm}
\end{figure}

* The reduced chord length $l_r$ is closely related to the (average) lengths $l_1$ and $l_2$ of the chords crossing phase 1 and phase 2, respectively in the arbitrarily chosen direction ($l_r^{-1} = l_1^{-1} + l_2^{-1}$) [5]