RECONSTRUCTION OF THE SCHRODINGER-EQUATION POTENTIAL FROM SCATTERING DATA BY THE KREIN METHOD

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Reconstruction of the Schrödinger-equation potential (for the case of s-scattering) from scattering data by the Krein method is discussed. Analytically, the problem reduces to the solution of a system of linear inhomogeneous algebraic equations for certain functions. The Bargmann potentials, determined earlier by other methods, are shown to result from the solution of the problem for various particular cases.

§1. INTRODUCTION

The inverse problem in scattering theory can be solved by one of three methods: the Gel’fand-Levitan method [3], the Marchenko method [4], or the Krein method [5-8]. The first two methods are the ones which have been primarily used in the extensive literature on this problem for various types of scattering (see the review [9]). Blažek has used the Marchenko method [10], and he has also used the Gel’fand-Levitan method to obtain Bargmann potentials [2].

In this paper, we will use the Krein method to reconstruct the potential.

The basic results in [5-8] can be formulated in the following manner. We write the Schrödinger equation for s-scattering

$$\psi'' + (\kappa^2 - V(r)) \psi(\kappa, r) = 0$$

(1.1)

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with the boundary conditions

$$\psi(\kappa, 0) = 0, \quad \psi'(\kappa, 0) = 1.$$  

(1.2)

The potential V(r) satisfies the condition

$$\int_0^\infty |V(r)| dr < \infty.$$  

(1.3)

The solution of (1.1), (1.2) can be written

$$\psi(\kappa, r) = \frac{\sin \kappa r}{\kappa} + \int_0^r \left[ \Gamma_+(r-u) - \Gamma_+(r+u) \right] \frac{\sin \kappa u}{\kappa} du,$$  

(1.4)

where the function \( \Gamma_+(t) \) is the solution of the integral equation

$$\Gamma_+(t) + \int_0^t H(t-u) \Gamma_+(u) du + H(t) = 0 \quad (0 \leq t \leq a)$$  

(1.5)
in which the kernel $H(t)$, an even function, is called an "accelerant." Using the function

$$A(r) = 2 \Gamma_2(2r),$$  \hspace{1cm} (1.6)

we can calculate the potential $V(r)$ from

$$V(r) = A^2(r) - A'(r).$$  \hspace{1cm} (1.7)

The spectral function $d\tau(\kappa)/d\kappa$ for the problem is related to the accelerant $H(t)$ by

$$H(t) = \frac{1}{2} \int_0^\infty \frac{\cos \kappa^2 t}{\kappa^2} d\tau(\kappa).$$  \hspace{1cm} (1.8)

When condition (1.3) holds, the spectral function is found from scattering data in the following manner:

$$\frac{d\tau(\kappa)}{d\kappa} = \begin{cases} \left( \frac{2\kappa^3}{\pi} \left( \frac{1}{|f(\kappa)|^2} - 1 \right) \right) (\kappa^2 \geq 0), \\ \left( \sum_{j=1}^N \frac{2\kappa^2 \kappa^2 (\kappa^2 + \kappa^2)}{\kappa^2} \right) \kappa^2 < 0, \end{cases}$$  \hspace{1cm} (1.9)

where $f(\kappa)$ is a Jost function, $E_j = -\kappa^2_j$ are the bound-state energies, and $c_j = \left[ \int_0^\infty \frac{\kappa^2 (i\kappa_j, \kappa) d\kappa}{\kappa^2} \right]^{-1}$ are the bound-state normalization constants.

Substituting (1.9) into (1.8), we find

$$H(t) = \frac{1}{\pi} \int_0^\infty \left( \frac{1}{|f(\kappa)|^2} - 1 \right) \cos \kappa^2 t d\kappa + \sum_{j=1}^N \frac{c_j}{2\kappa^2_j} \chi \kappa^2_j.$$  \hspace{1cm} (1.10)

Following Blažek [2], we write the Jost function as

$$f(\kappa) = \prod_{j=1}^N \frac{\kappa + i\kappa_j}{\kappa + i\kappa_j},$$  \hspace{1cm} (1.11)

where $\kappa_j$ and $\kappa_j$ are real nonzero numbers. The numbers $k = i\kappa_j$ ($\kappa_j > 0$) give the bound-state energies $E_j = -\kappa_j^2$.

We assume that $\kappa_1, \kappa_2, \ldots, \kappa_N (0 \leq n \leq N)$ are the numbers corresponding to bound states. Using (1.11), we can rewrite the accelerant (1.10) as

$$H(t) = \sum_{j=1}^N \frac{A_j}{2\kappa_j} e^{-\kappa_j t} + \sum_{j=1}^N \frac{c_j}{2\kappa_j} \chi \kappa_j t^2 (0 \leq n \leq N),$$  \hspace{1cm} (1.12)