PLANAR SURFACE-WAVE WAVEGUIDES
OF FINITE WIDTH

V. L. Mironov and V. V. Shevchenko

Dispersion equations for scalar (acoustic) surface waves propagating along an impedance band on an absolutely rigid plane and along an impedance tape are derived and analyzed.

Theoretical analysis of planar open structure capable of channeling surface waves in both acoustics [1] and electrodynamics [2] is usually based on the assumption that the corresponding systems are infinite and uniform along a coordinate perpendicular to the wave-propagation direction. In this paper we study planar acoustic surface-wave waveguides of finite width, in the description of which we use the concept of surface impedance.

**Impedance Band**

We assume that on the \( x = 0 \) plane there is a band \( |y| \leq a \) which has a surface admittance \(-iY\). To the left \( (x = 0, y < -a) \) and to the right \( (x = 0, y > a) \) on the band, the admittance of the plane is \(-iY_1\) and \(-iY_2\), respectively; here we have \( Y_1 < Y \) and \( Y_2 < Y \).

A surface wave on an infinite impedance plane with an admittance \(-iY\) incident obliquely on a linear admittance cut \((\Delta Y_j = Y - Y_j, j = 1, 2)\) is known [3] to undergo total reflection in the incidence-angle interval \((\alpha_j) = \arcsin (v_j/v) < \alpha < \pi/2, v_j = \sqrt{1 + Y_j^2}, v = \sqrt{1 + Y^2}\); the incidence angle \(\alpha\) is reckoned from the normal to the cut. Far from the cut we have the following asymptotic estimate for the residual field (after the surface-wave field is subtracted) in the angular interval \(\alpha_i = \arcsin (1/v) < \alpha < \pi/2\):

\[
U_r \sim e^{-m\pi i/\sqrt{kmr}},
\]

where \(m = \sqrt{\nu^2 \sin^2 \alpha - 1}\), \(r\) is the distance from the cut; and \(\kappa\) is the wave number in the \(x > 0\) space.

Neglecting the interaction of reactive fields (1), which are concentrated near the admittance cut, we assume that the field of natural waves of the impedance band is formed as a result of the successive re-reflection of plane surface waves from the admittance cuts. Then the acoustic potential for the interacting part of the field of natural waves can be written as

\[
U_y = (A e^{ig} + B e^{-ig}) e^{ikx - \kappa y},
\]

where \(g = \kappa \cos \alpha; h = \sqrt{\nu^2 - g^2} = \kappa \sin \alpha\). The unknown amplitudes \(A\) and \(B\) are determined from the requirement that the wave field coming from the boundaries of the band be equal to the field of the incident wave multiplied by the corresponding reflection coefficient in the \(y = \pm a\) planes

\[
Be^{-i\kappa a} = AR_1(g) e^{i\alpha a}, \quad Ae^{-i\kappa a} = BR_1(g) e^{i\alpha a},
\]

where \(R_j(g)\) is the coefficient for the reflection of a partial wave from the admittance discontinuity \(\Delta Y_j\).

The consistence condition for system (3) leads to a characteristic equation for natural waves of the band:

\[
e^{-i\kappa a} = R_1(g) R_2(g).
\]
For a symmetric structure \( Y_1 = Y_2 \) Eq. (4) simplifies to
\[
e^{-\mu x a} = \pm R_1(g). \tag{5}
\]

The coefficient for the reflection of a plane surface wave from an admittance discontinuity \( \Delta Y \) was found in [3] for the angular interval \( (\alpha_2)_1 < \alpha < \pi/2 \). Rewriting this result somewhat, we find
\[
R_1(g) = \exp \left[ -i \left( \frac{\pi}{2} - 1 \right) \frac{n_1}{n'} \ln \frac{Y + n}{Y - n} + \arctg \frac{n_1}{n} \right]
\]
\[
- \frac{1}{\pi} \left[ \frac{\pi}{2} \frac{\mu}{2} \arctg \frac{1}{\sqrt{\mu^2 - m^2}} - \frac{n_1}{m} \int \frac{\cos \varphi}{\sqrt{\varphi^2 + m^2}} d\varphi \right]
\]
\[
\frac{1}{\pi} \left[ \frac{\pi}{2} \frac{\mu}{2} \arctg \frac{1}{\sqrt{\mu^2 - m^2}} - \frac{n_1}{m} \int \frac{\cos \varphi}{\sqrt{\varphi^2 + m^2}} d\varphi \right]
\]
where
\[
n = \frac{g}{\kappa} = \nu \cos \alpha = \sqrt{Y^2 - m^2}, \quad n_1 = \sqrt{m^2 - Y_1^2}.
\]

Let us analyze the particular case of \( Y_1 = 0 \) in more detail. Substituting Eq. (6) with \( Y_1 = 0 \) into Eq. (5), we find a dispersion equation for an impedance band on an absolutely rigid screen:
\[
2 \kappa a Y = \frac{\pi}{2} + \arctg \frac{1}{\sqrt{\kappa^2 - m^2}} - \frac{1}{\pi} \int \frac{\mu}{\pi} \frac{2}{\sqrt{\mu^2 - m^2}} d\varphi + \mu \pi.
\]
\[
\frac{1}{\pi} \left[ \frac{\pi}{2} \frac{\mu}{2} \arctg \frac{1}{\sqrt{\mu^2 - m^2}} - \frac{n_1}{m} \int \frac{\cos \varphi}{\sqrt{\varphi^2 + m^2}} d\varphi \right]
\]
where \( \kappa = \frac{\nu}{\mu} \) is an unknown. The values \( \mu = 0, 2, 4, \ldots \) and \( \mu = 1, 3, 5, \ldots \) correspond to cosinusoidal (even waves) and sinusoidal (odd waves) potential distributions (2) along the oy axis.

The critical partial-wave incidence angle \( \alpha = \alpha_1 \), which coincides in this particular case \( Y_1 = 0 \) with the total-reflection angle, evidently gives a critical lower limit on the waveguide size. Substituting \( \kappa = 1 \) into Eq. (7), corresponding to the case \( \alpha = \alpha_1 \), we find
\[
\alpha_c = \frac{\pi}{2} \left( \frac{\mu + 1}{4} \right).
\]

Accordingly, for a fixed frequency, the lower limit on the band with range in which surface waves can be channeled is inversely proportional to the modulus of the admittance and directly proportional to the wave number.

Equation (7) has a simple solution in the particular case \( \kappa < 1 \). Expanding the second and third terms on the right side of Eq. (7) in series in powers of \( \kappa \), and retaining only terms of a first order of smallness, we find
\[
\frac{2 \kappa a Y}{\kappa} = \frac{\pi}{2} + \arctg \frac{1}{\sqrt{\kappa^2 - m^2}} - \frac{1}{\pi} \int \frac{\mu}{\sqrt{\kappa^2 - m^2}} d\varphi + \mu \pi.
\]
\[
\frac{1}{\pi} \left[ \frac{\pi}{2} \frac{\mu}{2} \arctg \frac{1}{\sqrt{\mu^2 - m^2}} - \frac{n_1}{m} \int \frac{\cos \varphi}{\sqrt{\varphi^2 + m^2}} d\varphi \right]
\]
We see from Eq. (8) that as \( \kappa a Y \to \infty \) we have the asymptotic relation
\[
g_{\alpha} a = (\mu + 1) \frac{\pi}{2},
\]
which corresponds to a waveguide with ideally soft walls in the \( y = \pm a \) planes.

The solid curves in Fig. 1 show the solutions of Eq. (7) for the first even \( (\mu = 0) \) and odd \( (\mu = 1) \) waves (curves 1 and 2); also shown here, by dashed curves, are the values found from Eq. (6).

It follows from (1) that the effect of the reactive fields which arise during the reflection of partial waves from the band boundary can be neglected under the condition
\[
2 \kappa a \gg 1.
\]