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ANALYTIC SOLUTIONS OF PARABOLIC AND
HYPERBOLIC HEAT-TRANSFER EQUATIONS
FOR NONLINEAR MEDIA
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New classes of analytic solutions are obtained which describe unsteady temperature distribu-
tions and take account of the temperature dependence of the thermophysical properties of the
material. The concept of a solution of the boundary layer transition type is introduced for the
generalized heat-transfer equation.

We consider the nonlinear heat-conduction equation in a one-dimensional plane region

\[ c(T) \partial T / \partial t = \lambda(T) \partial^2 T / \partial x^2. \]  

(1)

We introduce a new function \( \xi = \xi(x, t) \) with the following properties:

\[ \xi_x = U(T), \quad \xi_t = \lambda T_x, \]
\[ U(T) = U_0 + \int_0^T c(T) dT, \quad U_0 = \text{const}. \]

We change from the variables \( (x, t) \) to new independent variables \( (\xi, t) \):

\[ d\xi = U(T) dx + (\lambda \xi T) dt, \]
\[ D(\xi, t) D(x, t) = U \neq 0, \]

so that the initial Eq. (1) takes the form

\[ \beta(T) \partial T / \partial t = [\lambda(T) \partial^2 T / \partial \xi^2]_{\xi}, \quad \beta = cU^{-\beta}, \quad T = T(\xi, t), \]

(2)

where the Cartesian coordinate is related to the new variable by the equation

\[ x(\xi, t) = \int_0^t \frac{d\xi}{U(\xi, 0)} - \int_0^t \lambda T \xi dt, \quad U = U[T(\xi, t)]. \]

(3)

A comparison of Eqs. (1) and (2) shows that to each one-dimensional unsteady temperature distribution in a
medium with the thermophysical parameters \( c(T) \) and \( \lambda(T) \) there corresponds a certain one-dimensional un-
steady temperature distribution in a medium with volumetric heat capacity \( \beta(T) \) and a thermal conductivity

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Consequently, different temperature distributions in different media are equivalent from the point of view of the transformation considered. Thus, it is possible to construct new classes of temperature distributions based on known exact and approximate solutions of the nonlinear heat-conduction equation. The boundaries of the spatial region in which the old solution was obtained are transformed by Eq. (3).

We note, e.g., that if \( \beta (T) = \text{const} \)

\[
e(\xi) = \beta \left( \frac{\beta T - 1}{U_0} \right)^{-2}.
\]

We present an example of the solution of the nonlinear hyperbolic heat-transfer equation in a semi-infinite region with a moving boundary. We take the generalized heat-transfer equation [1–3] in the form

\[
c \dddot{T} + \gamma \ddot{T} + \dot{q}_v = (\lambda T_x)_x
\]

We consider a temperature interval \( T \in (0, T'] \) in which the thermophysical properties of the material vary as follows:

\[
c(T) = c_0 + c_1 T, \quad \lambda(T) = \lambda_0 + \lambda_1 T, \quad c_i, \lambda_i = \text{const}, i = 0, 1,
\]

\[
\gamma(T, t) = \gamma_0 + \gamma_1(t) T, \quad \gamma_0 = \text{const}, \gamma_1 \geq 0.
\]

The internal heat source strength is \( \dot{q}_v = \sum_{i=1}^5 q_i(t) T^i \). In the class of solutions given below we assume that if the value \( \gamma = 0 \) is not excluded in considering heat transfer in the temperature range under study, \( q_2 \neq 0 \). If \( \gamma > 0 \) everywhere, we can take \( q_2 = 0, q_v = 0 \).

The initial and boundary conditions are:

\[
t = 0: T(x, 0) = T^{(0)}(x), \ T_t(x, 0) = T^{(1)}(x); \ 0 \leq t < \infty, \quad x \to -\infty, T \to 0; \ x = x_b(t), \ T = T_b(t), \ -\infty < x < x_b(t).
\]

Equation (4) is satisfied by the following expressions:

\[
T = S(T), \ \lambda T_x = \xi + \eta, \ \eta = q_v + c \gamma T, \ \gamma = \eta(x, t),
\]

\[
D(\xi, t) = \frac{c(T) dT}{S(\xi, 0)}.
\]

We introduce new independent variables \((\xi, t)\):

\[
d \xi = S \dot{x} + (\lambda S T_x - \eta) dt,
\]

\[
D(\xi, t)/D(x, t) = S = \frac{\dot{S}}{S}, \ 0 < T < T',
\]

\[
x(\xi, t) = \int_0^t \frac{d \xi}{S(\xi, 0)} - \int_0^t \frac{\lambda S T_x - \eta}{S} dt.
\]

and instead of (4) we obtain for \( T(\xi, t) \) and \( \eta(\xi, T) \):

\[
(c_0 + c_1 T)(T_t - \eta T_x) + \dot{S} \eta = \frac{c(T)}{S} \left( \lambda T_x + \lambda_1 T_x^2 \right).
\]

\[
S \eta = q_v + c \gamma \left[ T_u + 3 \lambda ST_x T_x - 2 \eta T_x - \eta T_x + \frac{d(\lambda S)}{dT} T_x^2 \right].
\]

Equation (6) is hyperbolic for \( \gamma > 0 \) if \( S \eta_\xi \) is replaced by (7).

We seek the solutions of (6) and (7) in the form

\[
T(\xi, t) = \sum_{n=1}^{\infty} T_n(t) \xi^n, \ \eta(\xi, t) = \sum_{n=1}^{\infty} \eta_n(t) \xi^{n+1}, \ \xi \in (0, \xi_0].
\]

Substitution of these series into the equations gives