Analytic formulas are derived for the effective complex dielectric constant of a matrix system with Maxwell-Wagner losses. An approximate solution is given for a system of dielectrics with a single relaxation time. It is shown that this approximation is applicable to systems containing a semiconductor with blocking electrodes. The effect of spread in the dimensions of the semiconductor component on the frequency dependence of losses in a matrix system is discussed.

INTRODUCTION

The dielectric properties of heterogeneous systems have been studied since Maxwell [1] and Wagner [2]. Despite the fact that a host of approximation formulas (see [3], for example) have been derived for the effective static dielectric constant $\varepsilon_s$, the dielectric losses in such systems have received much less attention. A very general calculation method is the use of the formalism of the complex dielectric constants (CDC). All the quantities $\varepsilon_k$ are assumed to be complex,

$$e_k = e_k^r - ie_k^i,$$  \hspace{1cm} (1)

and only conductance of the components of the system is taken into account:

$$e_k = \frac{\sigma_k}{\omega}.$$  \hspace{1cm} (2)

Then, having substituted complex values for all of the values in the corresponding formulas for $\varepsilon_s$, according to (1), we separate the real and imaginary parts, and we obtain explicit expressions for the effective $\varepsilon^r$ and $\varepsilon^i$ for the system as functions of the parameters of the components. Unfortunately, this operation is often too time-consuming. It should also be borne in mind that the formulas for $\varepsilon_s$ sometimes contain nonlinear operations. The well-known formulas of Bruggeman [4] provide an example, and their use in CDC formalism requires care.

§1. TWO-LAYER DIELECTRIC

A two-layer dielectric is an elementary model of a heterogeneous system that permits an exact solution (Fig. 1a) [1, 2, 5]. It can be seen that this solution is also given by the CDC formalism.

Let $\varepsilon_1 = \varepsilon_1^r - i\varepsilon_1^i$ and $\varepsilon_2 = \varepsilon_2^r - i\varepsilon_2^i$ be the complex dielectric constants of the components; let $\gamma_1$ be the volume concentration of the first component in the system; and let $\omega$ be the angular frequency of the electric field that is applied to the system. The expression for $\varepsilon_s$ has the form [6]

$$\varepsilon_s = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 (1 - \gamma_1) + \varepsilon_2 \gamma_1}.$$ \hspace{1cm} (3)

Calculation according to the CDC formalism gives

$$\varepsilon^r = \varepsilon_\infty^r + \frac{\varepsilon_0^r - \varepsilon_\infty^r}{1 + (\omega \tau)^2},$$ \hspace{1cm} (4)

$$\varepsilon^i = \varepsilon_\infty^i + \frac{\varepsilon_0^i - \varepsilon_\infty^i}{1 + (\omega \tau)^2},$$ \hspace{1cm} (5)

where

$$\varepsilon_0^r = \frac{\varepsilon_0^r \gamma_1 + \varepsilon_1^r (1 - \gamma_1)}{[\varepsilon_0^r (1 - \gamma_1) + \varepsilon_1^r \gamma_1]^2},$$ \hspace{1cm} (6)

$$\varepsilon_0^i = \frac{\varepsilon_0^i \gamma_1 + \varepsilon_1^i (1 - \gamma_1)}{[\varepsilon_0^i (1 - \gamma_1) + \varepsilon_1^i \gamma_1]^2},$$ \hspace{1cm} (7)

$$\varepsilon_\infty^r = \frac{\varepsilon_0^r \gamma_1 + \varepsilon_1^r (1 - \gamma_1)}{[\varepsilon_0^r (1 - \gamma_1) + \varepsilon_1^r \gamma_1]^2},$$ \hspace{1cm} (8)

$$\varepsilon_\infty^i = \frac{\varepsilon_0^i \gamma_1 + \varepsilon_1^i (1 - \gamma_1)}{[\varepsilon_0^i (1 - \gamma_1) + \varepsilon_1^i \gamma_1]^2}.$$ \hspace{1cm} (9)

In formula (9), $\tau_1$ and $\tau_2$ are the Maxwell time constants of the components, and $\tau$ is the same value for the system. If the losses are due only to the conductance ($\sigma_k$) of the components, then for all of the time constants

$$\tau_k = \frac{\varepsilon_k^r}{\sigma_k}.$$ \hspace{1cm} (10)

The formulas for $\sigma$ (the effective conductance of the system) and $\sigma_0$ are obtained from (5) and (8) by substituting $\sigma_k$ for $\varepsilon_k^r$, in accordance with (2). For $\sigma_\infty$ we have

$$\sigma_\infty = \frac{\sigma_0 \gamma_1 + \sigma_1 (1 - \gamma_1)}{[\sigma_0 (1 - \gamma_1) + \sigma_1 \gamma_1]^2}.$$ \hspace{1cm} (11)

It is also easy to verify that

$$\varepsilon_0^0 = \sigma_\infty - \sigma_0 \tau.$$ \hspace{1cm} (12)

Relations (1) and (2) for $\varepsilon^r$ and $\varepsilon^i$ of the system are perfectly analogous to the classical Debye relations for

---

*Read at the Second All-Union Conference on Electroluminescence, 3 July 1967.
a homogeneous dielectric with one relaxation time $\tau_p$ and direct conductance (see [7], for example). In a heterogeneous system, $\tau_p$ corresponds to $\tau$, which characterizes the redistribution time of the field between the layers [5].

The losses in the system are Maxwell-Wagner losses [7]:

$$\tan \delta = \frac{\varepsilon'}{\varepsilon''}. \quad (13)$$

Maximum losses correspond to the frequency

$$\omega^* = \frac{1}{2} \frac{\omega_0}{\varepsilon''} \sqrt{\frac{\varepsilon''}{\varepsilon_{\infty}} + 3 / \varepsilon'' \left(1 - 10 \varepsilon'' + 9 \varepsilon''^2\right)}, \quad (14)$$

where

$$\alpha = \frac{\varepsilon'' / \varepsilon_{\infty}}{\varepsilon'' / \varepsilon_{\infty}}, \quad \omega_0 = \frac{1}{\tau}. \quad (15)$$

On the basis of the above formulas, it is easy to see that $\varepsilon_{\infty} < \varepsilon_1$ and $\varepsilon'' < \varepsilon_S'$; therefore, $\alpha < 1$ (see also Figs. 2 and 3).

It follows from (14) that $\tan \delta$ has a maximum only if

$$\alpha < \frac{1}{9}, \quad (16)$$

since the case of $\alpha \geq 1$ does not occur for real systems.

Calculation of the breakdown characteristics requires an expression for the field strength $E_1$ within one of the components (see [8], for example), if the strength of the external homogeneous field $E = U/d$ is known ($U$ is the voltage applied to the system). A general relation for $f_1 = E_1/E$ in the static case is given in [3]:

$$f_1 = \frac{\varepsilon - \varepsilon_{\infty}}{(\varepsilon_1 - \varepsilon_{\infty}) \gamma_1}. \quad (17)$$

If we include (17) in the CDC formalism, we find for the two-layer system

$$f_1 = A \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2}{1 + (\omega \tau)^2}}, \quad (18)$$

where

$$A = \frac{1}{\varepsilon_1 (1 - \gamma_1) + \varepsilon_2 \gamma_1}. \quad (19)$$

At frequencies $\omega \gg \omega_0$

$$f_{10} = \frac{\varepsilon_{\infty}}{\varepsilon_1}, \quad (20)$$

and at low frequencies ($\omega \ll \omega_0$)

$$f_{10} = \frac{\varepsilon_1}{\varepsilon_1}. \quad (21)$$

The qualitative variation of the effective parameters of the system as functions of frequency is shown in Figs. 2 and 3; this also explains the physical meaning of the symbols. At extremal frequencies, $\varepsilon'$, $\sigma$,

and $f_1$ approach their limiting values. The curve of $\tan \delta$ (curve 3, Fig. 2) passes through a maximum at $\omega^* > \omega_0$. The increase in losses at low frequencies is due to direct conductance. But if $\alpha > 1/9$, the losses do not have a maximum, and are similar to direct-conductance losses in a homogeneous dielectric (curve 4, Fig. 2). The shape of the curves $f = f(\omega)$ (Fig. 3) clearly reflects the fact that the field is redistributed in accordance with the conductances of the components at low frequencies (formula (21)) and in accordance with their dielectric constants at high frequencies (formula (20)). Field redistribution occurs in the frequency range $\omega = 1/\tau$.

§2. MATRIX SYSTEM

Many real systems can be described by a model in which one of the components surrounds unextended inclusions of a second component that are uniformly distributed in it (Fig. 1b)—a so-called matrix system [6]. For $\varepsilon_S$ of such a system, we have the expression

$$\frac{\varepsilon_S - \varepsilon_2}{\varepsilon + 2 \varepsilon_2} = \gamma_1 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \varepsilon_2}, \quad (22)$$

where the subscript 1 refers to the material of the unextended inclusions. Katona [9] used the CDC formalism to calculate $f_1$ of such a system assuming $\varepsilon_S^* = 0$ and calculated the resultant cumbersome expression on a computer. He did not give $f_1$ for the complete system ($\varepsilon_S^* = 0$) or expressions for $\varepsilon'$ and $\varepsilon''$.

If we make a simple but time-consuming CDC-formalism calculation on the basis of (22), we see that relations (4), (5), (12), (14), (15), (16), and (18) are also valid for a matrix system. This means that the frequency dependences of the effective parameters of a