States of a relativistic electron having an anomalous magnetic moment and moving in a centrally symmetric field are classified by a method most similar to a nonrelativistic classification. A spin quantum number is clearly defined. The motion of a neutron in a central field is also investigated. Some new, exact solutions of the Dirac–Pauli equation are found. A classical interpretation of the results is conducted.

The problem of the motion of an electron in a centrally symmetric field in relativistic quantum mechanics has been examined repeatedly and is discussed in many textbooks. However, the classification of the states of a relativistic electron differs noticeably from the corresponding nonrelativistic one [1], which presents definite disadvantages. It is true that in [2], an attempt was made at a new classification, but maximum total agreement with the nonrelativistic theory was not attained.

The influence of vacuum effects on the motion of an electron in a central field is usually studied by the straightforward techniques of quantum electrodynamics [1], where the calculations are complicated and unclear. However, there exists the possibility, indicated by Pauli [3, 4], of taking into account that part of the vacuum effects due to the existence of the anomalous magnetic moment of the particle by the addition of correction terms to the Dirac Hamiltonian.

In the present paper, it is shown that one can carry out that classification of the states of a relativistic fermion, having an anomalous magnetic moment and moving in a centrally symmetric, electrostatic field, closest to a nonrelativistic classification. Calculation of a Pauli term (anomalous magnetic moment) results in the removal of spin degeneracy. The motion of an uncharged fermion with an anomalous magnetic moment (neutron) is also considered and several exact solutions for centrally symmetric fields of various configurations are found. The possibility for the existence of bound states for a neutron in such field is revealed.

1. The Dirac–Pauli Equation for a Fermion in a Centrally Symmetric Field and the Classification of States

We shall describe the motion of a negatively charged fermion (charge e, rest mass m₀), possessing an anomalous magnetic moment μ, in a spherically symmetric electrostatic field defined by a potential A₀(r) and a field intensity E = -A₀(r) r/r, by the time-dependent Dirac–Pauli equation [3, 4]

\[ Hψ(r) = \hat{c}h\dot{ψ}(r), \quad H = c(xp) + \frac{p^2}{2m_0} - eA_0 + \frac{\mu}{c}aE. \] (1)

In the nonrelativistic treatment of the problem (disregarding the anomalous magnetic moment μ, of course), it is usual to select in addition to the total energy \( E = cK \), the projection of the angular momentum on the Z axis and the square of the angular momentum as a complete set of integrals of motion.

These same integrals of motion have a place also in the relativistic theory when \( \mu = 0 \). It is found that when \( \mu \neq 0 \) the situation is unchanged. Actually, it is easy to verify by direct inspection that the \( J_z \)
and $J^2$ operators, commuting with each other, commute also with the Hamiltonian (1); at the same time, as is usual, by $J$ we shall understand the total angular momentum

$$J = L + \frac{\hbar}{2} \sigma, \quad L = [rp].$$

Thus, the wave function $\psi(r)$ can be selected as a general function of the $H$, $J_z$, $J^2$ operators and the corresponding quantum numbers have a direct nonrelativistic analogy. However, these three integrals of motion do not determine unambiguously the wave functions in a relativistic theory. There exists a fourth integral of motion, proposed by Dirac, commuting with the preceding three,

$$S = \rho_3 \left( \pi J \right) - \frac{\hbar}{2}.$$

The $S$ operator describes the orientation of the spin of the particle. In fact, since the spin states of a particle can be explicitly distinguished only in a nonrelativistic approximation, then in order to clarify the meaning of $S$, it is necessary to construct its nonrelativistic limit. This is easy to do, observing that in the nonrelativistic approximation $\sigma$ and $J$ do not change, but $\rho_3 \to 1$. Consequently,

$$S \to \pi J - \frac{\hbar}{2}.$$

The constant $\hbar/2$ is unimportant and we see that $S$ has the sense of a projection of the spin on the angular momentum of the particle. This interpretation of the $S$ operator is also confirmed by the existence of the exact equation

$$S^2 = J^2 + \frac{\hbar^2}{4}. \quad (2)$$

It is obvious that Eq. (2) also relates the eigenvalues of the $S$ and $J^2$ operators. The eigenvalues of the $J_z$ operator are equal, as is known, to $\hbar j_z$, while for the $J^2$ operator they are equal to $\hbar^2 j(j + 1)$, where $j_z$ and $j$ are half-integers. We shall assume that $j_z = m - 1/2$, $j = l - 1/2$, where $m$ and $l$ are integers and are such that $l = 1, 2, 3... l \geq m \geq -l + 1$,

i.e., a specified value of $l$ corresponds to $2l$ states with different values of $m$ (but not $2l + 1$, as occurred in the nonrelativistic theory). Consequently, in the relativistic theory with a specified square of the total momentum there is an even number of states distinguished by the $Z$ components of the angular momentum. There is an odd number of such states in the nonrelativistic theory. It follows from Eq. (2) that the squares of the eigenvalues of the $S$ operator equal $\hbar^2 l^2$. Thus, the wave functions $\psi(r)$ can be completely determined from the solution of Eq. (1) and the supplementary equations

$$J_z \psi(r) = \hbar \left( m - 1 \right) \psi(r), \quad J^2 \psi(r) = \hbar^2 \left( l^2 - \frac{1}{4} \right) \psi(r), \quad S \psi(r) = \hbar l \psi(r), \quad \zeta = \pm 1,$$

where the quantum number $\zeta = \pm 1$ characterizes the two possible orientations of the spin of the particle ($\zeta = 1$ in the direction of the angular momentum, $\zeta = -1$ contrary to the direction of the angular momentum).

A common solution of Eqs. (1) and (3) can be written in the form

$$\psi(r) = \frac{1}{2r} \sqrt{\frac{(l - m)!}{\pi (l + m - 1)!}} \left[ \frac{1 + \zeta}{2} \psi_1(r) + \frac{1 - \zeta}{2} \psi_{-1}(r) \right],$$

$$\psi_1(r) = \left( \begin{array}{c} (m + l - 1) Y^{m-1}_{l-1} f_1(r) \\ Y^m_{l-1} f_1(r) \\ \iota (m + l - 1) Y^{m-1}_{l+1} g_1(r) \\ \iota Y^m_{l+1} g_1(r) \end{array} \right),$$

$$\psi_{-1}(r) = \left( \begin{array}{c} (m - l - 1) Y^{m-1}_{l-1} f_{-1}(r) \\ Y^m_{l-1} f_{-1}(r) \\ \iota (m - l - 1) Y^{m-1}_{l+1} g_{-1}(r) \\ \iota Y^m_{l+1} g_{-1}(r) \end{array} \right).$$

*The range of variation for $m$ follows from the obvious inequality $J_z^2 \leq J^2$. 

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