The possibility of obtaining new exact solutions of the Einstein equations in vacuum and of the Einstein-Maxwell equations from known solutions through a zero-coupling transformation is analyzed. The changes in the algebraic properties resulting from this transformation are analyzed for spaces which allow a congruence of isotropic geodesics with no distortion (shear) or rotation (curl). Examples of the derivation of new exact solutions are considered.

New methods for constructing exact solutions of the Einstein equations through the application of non-coordinate (e.g., conformal) transformations to the metrics of known solutions have been developed extensively [1-7]. These methods have also been used successfully to derive solutions for problems involving the presence of matter from solutions for the case of vacuum [1-4]. In particular, quite interesting results have been obtained through the use of a "zero-coupling" transformation of the type

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \kappa_{\mu} \kappa_{\nu}, \quad \kappa_{\mu} \kappa_{\nu} g^{\mu\nu} = \kappa_{\mu} \kappa_{\nu}, \quad g^{\mu\nu} = 0. \quad (1)$$

This transformation was studied in detail by Kerr and Schild [8] for the case in which the original metric $g_{\mu\nu}$ is the metric of a Minkowski space; this transformation was studied further in [9, 10].

We previously [11] analyzed solutions like (1) on the basis of the Newman-Penrose formalism [12] for the case in which the congruence of isotropic lines with a tangential vector field $\kappa_{\mu}$ is geodesic. To write the field equations in the Newman-Penrose formalism we use a quasiorthogonal tetrad of four isotropic vectors:

$$Z_{\mu\nu} = (l_{\mu}, m_{\mu}, \bar{m}_{\mu}, \bar{l}_{\mu}), \quad l_{\mu} n_{\mu} = -m_{\mu} \bar{m}_{\mu} = 1, \quad g_{\mu\nu} = n_{\mu} l_{\nu} + l_{\mu} n_{\nu} - m_{\mu} \bar{m}_{\nu} - \bar{m}_{\mu} m_{\nu}. \quad (2)$$

As vector $\kappa_{\mu}$ here we use the vector $l_{\mu}$ of this tetrad. By choosing the affine parameter along geodesics we can satisfy the relations $l_{\mu} ; v_{\mu} = 0$. In this case the transformation of type (1) becomes

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2a l_{\nu}, \quad \tilde{\kappa}_{\mu} = g^{\mu\nu} - 2a l_{\mu} l_{\nu}, \quad (3)$$

$$\tilde{n}_{\mu} = n_{\mu} + a l_{\mu}, \quad \tilde{l}_{\mu} = l_{\mu}, \quad \tilde{m}_{\mu} = m_{\mu},$$

where $a$ is some function of the coordinates. Here and below the symbol "~" shows that a quantity refers to "coupled spaces" with metric $\tilde{g}_{\mu\nu}$.

Since congruence $l_{\mu}$ is geodesic, we were able to prove the following important theorem [11]: all spaces of type (1) [or (3), which includes the original space] are algebraically special according to the Petrov classification, and vector $l_{\mu}$ is the double Debever-Penrose root

$$\Psi_0 = -C_{\nu\rho\sigma} b^\nu b^\rho b^\sigma = 0, \quad \Psi_1 = -C_{\nu\rho\sigma} l^\nu b^\rho b^\sigma = 0. \quad (4)$$

According to the Goldberg-Sachs theorem, an isotropic congruence with a tangential vector $l_{\mu}$ in such spaces has no distortion ($\sigma = 0$).

The original space $V_4$ is assumed empty, and the coupled space $\tilde{V}_4$ contains an electromagnetic field (or is also empty); in terms of the Newman-Penrose formalism, we would write...
We also require that the other vectors of the tetrad propagate in a parallel manner along the geodesic congruence. Then the following conditions hold:

\[ \kappa = \varepsilon = \Pi = \sigma = 0, \quad \Psi = \Psi = \tilde{\Psi} = \tilde{\Psi}_1 = 0. \]

As a result of transformation (3) the Ricci rotation coefficients change in the following manner:

\[ \kappa = \tilde{\kappa} = \varepsilon = \tilde{\varepsilon} = \sigma = \tilde{\sigma} = 0, \]

\[ \tilde{\kappa} = \lambda, \quad \tilde{\varepsilon} = \tau, \quad \tilde{\sigma} = \beta, \quad \tilde{\lambda} = \tilde{\tau} = \tilde{\beta} = \tilde{\sigma} = 0, \quad \tilde{\rho} = \rho, \quad \tilde{\sigma} = \sigma + a \rho, \quad \tilde{\rho} = \rho + a \sigma. \]

In studying the properties of transformation (3), we use the "difference equations" obtained by subtracting the Newman-Penrose equations for metric \( g_{\mu \nu} \) from the corresponding equations for metric \( \tilde{g}_{\mu \nu} \). Obviously, if metric \( g_{\mu \nu} \) is a solution of the Einstein equations in vacuum, the satisfaction of the difference equations would be a necessary and sufficient condition for the metric \( \tilde{g}_{\mu \nu} \) to be a solution of the Einstein-Maxwell equations (or of the Einstein equations in vacuum as a particular case). To obtain the difference equations we use the Newman-Penrose equations, Eqs. (4.2) in [12]. However, there are several misprints in [12] in the expression for the Bianchi identities in the presence of an electromagnetic field. The correct Bianchi identities in the Newman-Penrose formalism are given in Appendix 1. All the difference equations are given in Appendix 2 (the first seven equations are consequences of Eqs. (4.2) or [12], while the last six are consequence of the Bianchi identities).

Our purpose here is to study the general properties of transformation (3) and the possibility of obtaining new exact solutions of the Einstein equations from known solutions. For simplicity, we will usually treat the case in which the coupled space is also empty, i.e., the case

\[ \tilde{\Phi}_\rho = \Phi_\rho \neq 0, \quad \rho, \sigma = 0, 1, 2. \]

\[ \Phi_\rho = 0, \quad \tilde{\Phi}_\rho \neq 0, \quad \rho, \sigma = 0, 1, 2. \]

(5)

(6)

(7)

(8)

In the Newman-Penrose formalism the tetrad components of the Weyl tensor are written as linear combinations of the five complex scalars \( \Psi_0 \) to \( \Psi_4 \). The zeroth tetrad can be chosen in such a manner that the \( \Psi \) assume canonical form for each Petrov type [13] (Table 1).

Here we choose the tetrad in such a manner that we always have \( \Psi_0 = \Psi_1 = 0 \). In general, we therefore have the correspondence shown in Table 2 between the type of space and the noncanonical form of the Weyl tensor.

Let us assume that we know the algebraic type of original space \( V_4 \), i.e., that the Weyl tensor can be written as one of the possible combinations in Table 2. Because of transformations (3), we generally find the most arbitrary set of scalars \( \Psi_0 \) to \( \Psi_4 \). To determine the type of the coupled space \( V_4 \), we must use orthogonal transformations of the tetrad which do not disrupt the type of metric. Of particular concern are the following:

a) two-parameter null rotations around \( l^\mu \),

\[ l^\rho = l^\rho, \quad m^\rho = m^\rho + p l^\rho, \quad n^\rho = n^\rho + p m^\rho + \bar{p} l^\rho; \]

(9a)

b) two-parameter null rotations around \( n^\mu \),

\[ l^\rho = l^\rho + q m^\rho + \bar{q} m^\rho + q n^\rho, \quad m^\rho = m^\rho + q n^\rho, \quad n^\rho = n^\rho. \]

(9b)

1515