A transformation of the Kerr–Schild zero-coupling type is used to obtain new exact solutions of the Einstein–Maxwell equations from the known steady-state solutions in vacuum, including the Kerr solution describing the gravitational field of a uniformly rotating ring.

There have always been certain difficulties involved in obtaining new exact solutions of the Einstein equations and, in particular, in interpreting them. Various noncoordinate transformations, which can be used to obtain new exact solutions from known solutions, can be of some assistance here. In this continuation of the study of [1, 2], we consider the application of the zero-coupling transformation [3]

\[ g_{\mu\nu} = g_{\mu\nu} + a_{\mu} l_{\nu}, \quad l_{\mu} l_{\nu} g_{\nu\lambda} = l_{\mu} l_{\nu} \tilde{g}_{\nu\lambda} = 0 \]  

(1)

to spaces \((g_{\mu\nu})\) which allow the existence of a congruence of isotropic geodesic lines without distortion (shear) but with rotation (curl) and divergence. As previously, we use the Newman–Penrose tetrad formalism [3], which displays very clearly the algebraic structure of the solutions. The absence of distortion from the congruence of isotropic geodesics with a tangential vector field \(l_{\mu}\) corresponds to the conditions

\[ \rho = \tilde{\rho} = 0. \]  

(2)

Solutions of this type were obtained previously [5, 6]; of particular interest among these solutions is the Kerr solution [5], which describes the gravitational field of a uniformly rotating ring [8]:

\[ ds^2 = (r^2 + s^2 \cos^2 \theta)(d\phi^2 + \sin^2 \theta d\varphi^2) + 2 (du + s \sin^2 \theta d\varphi) \times (dr + s \cdot \sin^2 \theta d\varphi) - \left(1 - \frac{2mr}{r^2 + s^2 \cos^2 \theta}\right) (du + s \sin^2 \theta d\varphi)^2, \]  

(3)

where \(s\) is a real constant.

We turn now to the effect of transformation (1) on this solution. In the Newman–Penrose formalism, it is written as

\[ g_{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - n^\mu m^\nu - m^\mu n^\nu, \]

\[ l^\mu = \delta^\mu_0, \quad n^\nu = \delta^\nu_1 + U \delta^\nu_4, \quad m^\nu = \omega \delta^\nu_2 + \xi \partial_1, \]

\[ l_\mu = \delta^\mu_1 + s \sin^2 \theta \delta^\mu_4, \quad n_\mu = -U \delta^\mu_1 + \delta^\mu_3 - s \sin^2 \theta (U + 1) \delta^\mu_4, \]

\[ m_\mu = -\frac{1}{\sqrt{2}} (r + is \cos \theta) (\partial_0 + i \sin \theta \partial_1), \quad \chi^\mu = (u, r, \theta, \varphi), \]

\[ i, \kappa = (1, 3, 4), \]

\[ U = -\frac{1}{2} - \frac{1}{2} m (p + \tilde{p}), \quad \omega = \omega^0 \rho, \quad \xi = \xi^0 \rho, \]

\[ \xi^{03} = -\frac{1}{\sqrt{2}}, \xi^{04} = -\frac{i}{\sqrt{2} \sin \theta}, \quad \omega^0 = -\xi^0 = i \frac{s \sin \theta}{V^2}. \]

(4)

The subscript "O" indicates that the quantity does not depend on \(r\). The nonvanishing Ricci rotation coefficients and components of the Weyl tensor are

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\[ \rho = -\frac{1}{r + is\cos \theta}, \quad \chi = -\rho \cos \theta = -\frac{\rho \cos \theta}{2\sqrt{2} \sin \theta}, \quad \gamma = \frac{1}{2} m_0, \]

\[ \nu = \frac{1}{2} (\rho + \rho'), \quad \nu = \frac{i}{\sqrt{2}} s \sin \theta m_0, \quad (5) \]

\[ \Psi_0 = \Psi_1 = 0, \quad \Psi_2 = m_0^2, \quad \Psi_3 = \frac{3}{2} ism_0 \sin \theta, \quad \Psi_4 = -3\rho^2 s^2 m \sin^2 \theta. \]

By using the Newman–Penrose formalism we can very easily determine the algebraic type of this solution. It follows from the table in [2] that this solution can be only a Petrov type II or D solution. Since we have the auxiliary condition

\[ 3\Psi_4 \Psi_2 = 2\Psi_3, \]

this solution is of type D.

Let us consider transformation (1). The difference equations from [2] can be simplified considerably in this case; many of them are satisfied identically. The rest are

\[ DDA - (\rho + \rho') Da + a_0 \rho = 0, \quad \Phi_{11} = \frac{1}{2} (\rho + \rho') Da - \frac{1}{2} a(\rho^2 + \rho), \]

\[ \Phi_{12} = \frac{1}{2} (m - p) \rho - \frac{1}{2} a\Delta \rho + \frac{1}{2} \delta Da, \]

\[ \Phi_{22} = -2 \frac{\delta a}{\rho} D a - 2a \rho (\chi + \gamma) + a \rho^2 - 2a \rho, \]

\[ 2a \Psi_2 - \rho \Delta a + \delta \Delta a + a \rho^2, \]

\[ \tilde{\Psi}_2 - \tilde{\Psi}_2 = \rho Da + a_0 (\rho - \rho'), \quad \tilde{\Psi}_3 - \tilde{\Psi}_3 = \frac{1}{2} (\rho - \rho') \Delta a + \frac{1}{2} a \Delta \rho + \frac{1}{2} \delta Da, \]

\[ \tilde{\Psi}_4 - \tilde{\Psi}_4 = 2a \delta a + \delta \Delta a. \]

It follows from the first equation in (7) that

\[ a = c (\rho + \rho') + d \rho, \]

where c and d are functions of \( x^4 \) alone. The components of the Weyl and Maxwell tensors in space \( V_4 \) are now written as

\[ \tilde{\Psi}_2 = (m + 2c) \rho^2 + 2d \rho^2, \quad \tilde{\Psi}_2 = \tilde{\xi}^{10} c_{11} (2\rho^2 + \rho'), \quad \tilde{\xi}^{10} d_{11} 2\rho^2 + \frac{3}{2} \rho^4 (m + 2c) is \sin \theta, \]

\[ \tilde{\Psi}_4 = -3 \rho^2 s^2 m \sin^2 \theta + 2x \delta a - \delta \Delta a, \quad \tilde{\Psi}_4 = \rho^2 \rho^2 d, \]

\[ \tilde{\Phi}_{11} = \rho^2 \tilde{\xi}^{10} c_{11} + \rho^2 \tilde{\xi}^{10} d_{11} + 2a \delta \rho^2 \rho^2, \quad \tilde{\Phi}_{12} = \rho^2 \tilde{\xi}^{10} c_{11} - \tilde{\xi}^{10} \tilde{c}_{11}, \]

\[ \tilde{\Phi}_{22} = -2 \tilde{\xi}^{10} \tilde{c}_{11} - 2a \rho (\chi + \gamma) + a \rho^2 - 2a \rho, \]

\[ + 2a \Psi_2 - \rho \Delta a + \delta \Delta a + a \rho^2, \]

\[ \tilde{\Psi}_2 - \tilde{\Psi}_2 = \rho Da + a_0 (\rho - \rho'), \quad \tilde{\Psi}_3 - \tilde{\Psi}_3 = \frac{1}{2} (\rho - \rho') \Delta a + \frac{1}{2} a \Delta \rho + \frac{1}{2} \delta Da, \]

\[ \tilde{\Psi}_4 - \tilde{\Psi}_4 = 2a \delta a + \delta \Delta a. \]

To complete the solution we must require that the components of Maxwell tensor \( \Phi_m \)

\[ \Phi_{mn} = \Phi_{m} \Phi_{n} \]

satisfy the Maxwell equations, which in our case are

\[ D \Phi_1 = 2\Phi_1, \quad 2 \Phi_1 = 0, \quad D \Phi_2 - \tilde{\Phi} \Phi_1 = \rho \Phi_2, \quad 2 \Phi_2 - \Delta \Phi_1 = 2\rho \Phi_1 + 2 \tilde{\Phi} \Phi_1. \]

Assuming \( \Phi_1 \) and \( \Phi_2 \) to have the form

\[ \Phi_1 = e^{i \psi} b, \quad b = \bar{b}, \quad b^2 = d, \quad \Phi_2 = A \phi + B \phi^2 + C \phi^3, \]

we find the following conditions from (11)

\[ \tilde{\xi}^{10} (b_{11} - i \phi_{11} b) = 0, \quad B = \tilde{\xi}^{10} (b_{11} + i \phi_{11} b) e^{i \psi} = 2e^{i \psi} \tilde{\xi}^{10} b_{11}, \]

\[ C = 2 \phi^{10} e^{i \psi} b, \quad \phi^{10} A_{11} = 2 \phi^{10} A = (b_{11} + i \phi_{11} b) e^{i \psi}, \]

\[ \psi B_{11} - 2 \phi^{10} B = 2is \cos \theta (b_{11} + i \phi_{11} b) e^{i \psi}. \]

(12)