DETERMINATION OF THE THERMAL BOUNDARY CONDITIONS FROM NONSTATIONARY-TEMPERATURE MEASUREMENT DATA

V. I. Zhuk, S. A. Il' in, and D. N. Chubarov

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A method of constructing a solution of the converse heat-conduction problem is presented. The inverse operator is represented in explicit form with a regularization parameter which depends on the level of the error in the initial data.

In the majority of papers on the solution of ill-posed converse problems in heat conduction, the problem is reduced to setting up constructive (numerically realizable) methods of solving integral equations. However, these problems do not enable one to obtain the inverse operator in explicit form and lead to the solution of either linear algebraic systems or to finding the eigen vectors and numbers of matrices [1-3]. Below we describe a method of constructing the regularized inverse operator in explicit form, which enables one to obtain a constructive solution of the initial ill-posed problem as it applies to reconstructing the thermal boundary conditions from data obtained from measurements of the nonstationary temperatures in the constructional elements at a certain distance from the heated surface.

Suppose that in a heat experiment a model of a plane unbounded plate is set up, which is subjected to heating or cooling on one side and is thermally insulated on the other. From measurements of the temperature at different times at a point with coordinate \( x = x_1 \) it is required to determine the heat flux and the temperature on the surface \( x = x_0 \) subjected to heating or cooling. The thermal properties over the temperature range employed in the experiment are assumed to be constant. The mathematical formulation of the problem therefore has the following form:

\[
\begin{align*}
\frac{\partial^2 T(F_0, \xi)}{\partial \xi^2} & = \frac{\partial T(F_0, \xi)}{\partial F_0}, \\
T(0, \xi) & = 0; \quad \frac{\partial T(F_0, \xi)}{\partial \xi} \bigg|_{\xi = 0} = 0; \quad T(F_0, \xi) \bigg|_{\xi = 1} = T(F_0, \xi_0) ; \\
\frac{\partial T(F_0, \xi)}{\partial \xi} \bigg|_{\xi = 1} & = ?, \quad T(F_0, 1) = ?
\end{align*}
\]

The solution of problem (1) in Laplace-transform space can be written in the following form [4]:

\[
\tilde{q}(s, 1) = \frac{T(s, \xi_0) \sqrt{s \sinh s}}{\sinh s \xi_1}; \quad T(s, 1) = \frac{T(s, \xi_0) \cosh s}{\cosh s \xi_1}.
\]

The ill-posed nature of the problem in this case manifests itself in the fact that the transforms \( \sqrt{s \sinh s / \cosh s \xi_1} \) and \( \cosh s / \sqrt{s \sinh s \xi_1} \) with \( \xi < 1 \), due to the fact that they approach \( \infty \) as \( |s| \to \infty \), do not satisfy one of the transformation requirements and do not have originals in the form of ordinary functions. To obtain a solution which can be transformed we will use Eq. (1) and introduce into it a "regularizing source" which depends on \( T(F_0, \xi) \) and the parameter \( \beta \)

\[
\frac{\partial^2 T(F_0, \xi)}{\partial \xi^2} + \int_0 F_0 \exp \left[ -\beta (F_0 - \tilde{F}_0) \right] d\tilde{F}_0 = \frac{\partial T(F_0, \xi)}{\partial F_0}.
\]

It is obvious that \( \beta \exp[ -\beta (F_0 - \tilde{F}_0)] \to \delta (F_0 - \tilde{F}_0) \) when \( \beta \to \infty \), where \( \delta (F_0 - \tilde{F}_0) \) is the Dirac function, and if the term introduced into (3) has the form

for fairly large $\beta_0$ the problem would become similar to the quasitransformation method [5]. In this case $\beta = \beta_0$ and when $\beta_0 \to \infty$ the last term in (3) would approach zero. We apply a Laplace transformation to (3)

$$sT(s, \xi) - \frac{\partial T(s, \xi)}{s + \beta} = \frac{\partial^2 T(s, \xi)}{s^2}$$

when $T(0, \xi) = \left. \frac{\partial T(Fo, \xi)}{\partial Fo} \right|_{Fo=0} = 0$. The solution of (5), taking (1') into account, has the form

$$\tilde{q}(s, 1) = T(s, \xi_f) \sqrt{\frac{-s^2b}{s^2 + \beta}} \frac{sh \sqrt{\frac{s^2b}{s + \beta}}}{ch \sqrt{\frac{s^2b}{s + \beta}} \xi_f},$$

$$T(s, 1) = T(s, \xi_f) \frac{ch \sqrt{\frac{s^2b}{s + \beta}}}{ch \sqrt{\frac{s^2b}{s + \beta}} \xi_f}.$$  

Henceforth we will confine ourselves to obtaining a solution for $\tilde{q}(Fo, 1)$. Expression (6) can be converted as follows:

$$\tilde{q}(s, 1) = T(s, \xi_f) \sqrt{\frac{-s^2b}{s + \beta}} \left( \frac{sh \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f)}{ch \sqrt{\frac{s^2b}{s + \beta}} \xi_f} + th \sqrt{\frac{s^2b}{s + \beta}} \xi_f ch \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f) \right).$$  

It is obvious that

$$\lim_{\beta \to \infty} \tilde{q}(s, 1) = T(s, \xi_f) \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f),$$

which corresponds to the relationship between $\tilde{q}(Fo, 1)$ and $T(Fo, \xi_1)$ in transformation space when solving the initial system (1). However, whereas (2) cannot be transformed in the sense of obtaining the inverse operator that is bounded when $Fo = 0$, Eq. (8) can be so transformed. In this case

$$\tilde{q}(Fo, 1) = \int_{0}^{Fo} \frac{dT(Fo, \xi_f)}{dFo} \Psi(Fo - \tilde{Fo}) d\tilde{Fo}.$$  

The function $\Psi$ is determined by the values for $Fo - \tilde{Fo}$ of the original of the expression

$$\Psi(s) = \frac{1}{s} \sqrt{\frac{s^2b}{s + \beta}} \left( \frac{sh \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f)}{ch \sqrt{\frac{s^2b}{s + \beta}} \xi_f} + th \sqrt{\frac{s^2b}{s + \beta}} \xi_f ch \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f) \right).$$  

The original of the first term in (10) can be obtained by expanding $sh \sqrt{\frac{s^2b}{s + \beta}}(1 - \xi_f)$ in series

$$q_1(\theta) = \sum_{n=0}^{\infty} \frac{\beta^{n+1}(1 - \xi_f)^{2n+1}}{(2n+1)!} L_n(\beta \theta) \exp(-\beta \theta),$$

where $L_n$ are Laguerre polynomials of order $n$. To obtain the original of the second term we expand the hyperbolic tangent and cosine in series as in [6]