BOUNDARY-LAYER THEORY APPLIED TO HIGH INJECTION RATES

É. A. Gershbein and V. N. Filimonov

The equations of an isothermal laminar multicomponent boundary layer are solved numerically for both velocity and concentration profiles at high injection rates. The results are evaluated comparatively.

We consider the isothermal flow of an incompressible multicomponent gas in a laminar boundary layer with a negative pressure gradient and a high injection number \( \psi \sim \sqrt{\text{Re}(\rho v)_{\infty}/(\rho v)_{\infty}} \). The stream in the boundary layer with heavy distributed injection can be divided into two regions [1-7]: 1) the inner region adjacent to the solid surface, where viscosity effect are negligible (to the first approximation); and 2) the "uplifted" viscous region with a transition from the inner-layer flow mode to the ideal-gas flow mode outside the boundary layer.

Self-adjoint flow modes were studied in [1-5]. Nonself-adjoint solutions to the boundary-layer equations were analyzed in [6, 7]. In [6], moreover, an asymptotic solution has been obtained to the Prandtl equation for a homogeneous incompressible fluid. Asymptotic formulas have also been obtained in [7] for the friction coefficient, for the thermal flux and the diffusion current in the fluid mixture components at the solid surface, and equations have been derived describing the flow of a compressible multicomponent gas in the sublayer region of a boundary layer. In our study here these equations will be solved for the isothermal flow of a multicomponent gas. A numerical solution will be given to the problem of gas flow in boundary layers of a sphere and of a circular cylinder.

1. The equations of a laminar isothermal multicomponent boundary layer, in the Dorodnitsyn–Lies variables

\[
\xi = \int_0^x \mu \rho u \mathrm{d}x, \quad \eta = \frac{\mu u}{\sqrt{2\beta}} \int_0^y \rho \mathrm{d}y
\]

are [7]:

\[
(f_i m_i^*)_n + \beta (\rho_j \theta - f^*_i) = 2\beta (F_i^* \xi - \xi^* F_i^* m_i^*),
\]

\[
X_i^* = (f + 2\xi) \xi_i - 2\xi_i \xi_k (i = 1, \ldots, n - 1),
\]

\[
I (c_j m_j^*) = \sum_{j=1}^{m_j^*} \frac{m_j^*}{m_j} S_{ij} (c_i X_j - c_j X_i) \quad (i = 1, \ldots, n - 1),
\]

\[
\sum_{i=1}^{n} c_i = 0, \quad \sum_{i=1}^{n} X_i = 0.
\]

Here

\[
\rho v = -r^4 \left[ (f + 2\xi) \xi^* + 2\xi^* \eta^* \right] / \sqrt{2\beta};
\]
System (1.1) is solved for the following boundary conditions:
\[ f_0 \to 1, \quad c_i \to c_i(0) (i=1, \ldots, n) \quad \text{as} \quad \eta \to \infty, \]
\[ f + 2\xi f_\xi = \Phi(\xi), \quad f_\eta = 0, \]
\[ \Phi(\xi)(c_i - c_i(1)) = X_i (i=1, \ldots, n) \quad \text{for} \quad \eta = 0. \]

Here \( c_i^{(1)} \) denotes the mass concentration of the \( i \)-th component in the injected gas mixture and
\[ \Phi(\xi) = - \frac{r \sqrt{2\xi}}{\xi_x} (\rho v)_w. \] (1.3)

Relation (1.3) can be rewritten as
\[ f(\xi, 0) = -\alpha(\xi) = - \frac{1}{V_0^{(1)}} \int_0^{(1)} r (\rho v)_w dx. \] (1.4)

2. We consider now the asymptotic solution to Eq. (1.1) at high values of the injection number \( \phi \). It can be shown [7] that in this case near the solid surface Eqs. (1.1) reduce (in the first approximation) to
\[ \left( \psi \right) + 2\xi \psi \psi_\xi + \beta (\rho_v / \rho - \psi_\xi) = 2\xi \psi_\xi, \] (2.1)
\[ \left( \psi \right)_\xi + 2\xi \psi \psi_\xi = 2\xi \psi_\xi c_i (i=1, \ldots, n-1) \] (2.2)

with the boundary conditions at \( \xi = 0 \)
\[ \psi = 1, \quad \psi_\xi = 0, \quad c_i = c_i(0) (i=1, \ldots, n-1). \] (2.3)

Here
\[ \psi = f(f(\xi, 0), \xi = \eta/f(\xi, 0), \psi = \Phi(\xi)/f(\xi, 0). \] (2.4)

It will be assumed further that the medium is incompressible and that the concentrations \( c_i^{(1)} \) do not depend on \( \xi \):
\[ c_i^{(1)}(\xi) = \text{const} (i=1, \ldots, n-1). \] (2.5)

The solution to Eqs. (2.2) with conditions (2.3) will then be
\[ c_i(\xi, \xi) = c_i(0) (i=1, \ldots, n), \quad \rho(\xi, \xi) = \rho_w. \] (2.6)

Let us now analyze Eq. (2.1). We introduce here new variables
\[ Z = \psi \rho_w / \rho, \quad \varphi = \varphi(\xi, \xi). \] (2.7)

Then Eq. (2.1) and conditions (2.3), with (2.5) taken into account, become
\[ \psi \varphi Z_{\varphi} - 2\xi Z_\xi = 2\xi (Z - 1), \] (2.8)
\[ Z = 0 \quad \text{for} \quad \varphi = 1. \] (2.9)

It is not difficult to see that function \( Z \) has the simple physical meaning:
\[ Z = \rho_w \mu^2 / \rho \mu^2. \] (2.10)

The solution to Eq. (2.8) with condition (2.9) is
\[ Z = 1 - \exp \left( \int_0^\xi \frac{\beta}{Z^2} d\xi \right). \] (2.11)

Here function \( \tau(\xi, \varphi) \) is determined from the equation
\[ \xi \alpha^2(\xi) \varphi^2 = \tau \alpha^2(\tau), \] (2.12)