With respect to small perturbations, we examine the stability of a steady flow (with a gradient) of a non-Newtonian fluid obeying a rheological power law in a flat channel. We have found the neutral stability curves for various values of the exponent $n$ in the rheological law.

In this paper we will investigate the stability of a steady plane flow with a gradient for fluids obeying a rheological power law, for which the relationship between the deviator of the stress tensor $s_{ij}$ and the strain-rate tensor $f_{ij}$ (the rheological law) is written [1] in the form

$$s_{ij} = 2k_n \omega^{n-1} f_{ij}$$  \hspace{1cm} (1)

where $\omega = \sqrt{2f_{ij}f_{ij}}$. On the basis of the adopted terminology, media with $n > 1$ are referred to as dilatational fluids, while those with $n < 1$ are known as pseudoplastic. The case $n = 1$ corresponds to a Newtonian fluid.

From the equation of motion for the medium, written in the absence of body forces,

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial s_{ij}}{\partial x_j}$$  \hspace{1cm} (2)

($\rho$ is the density of the medium; $p$ is the pressure; $v_i$ is the component of the velocity vector) for a steady flow in a plane channel under the action of a constant pressure gradient in the direction of the axis $x_1 \equiv x$ ($v_1 = U$, $v_2 = v_3 = 0$) with consideration of the boundary conditions we can find the profile of the dimensionless velocity [2] in the form

$$U(y) = 1 - |y|^{\frac{n+1}{n}}$$  \hspace{1cm} (3)

with the axis $x_2 \equiv y$ perpendicular to the channel wall; in making the transition to the dimensionless quantities, we have taken the maximum velocity at the center of the channel for the case in which $y = 0$ as the characteristic velocity; we have taken the half-width of the channel as the characteristic dimension.

The stability of flow (3) is studied in relation to small two-dimensional perturbations in the velocities $u'$ and $v'$ along the $x$- and $y$-axes, respectively. The equations of motion and continuity are linearized in the usual manner [3]. If we introduce the stream function for the perturbations

$$u' = \frac{\partial \Psi}{\partial y}; \quad v' = -\frac{\partial \Psi}{\partial x}$$  \hspace{1cm} (4)

and seek the solution for $\Psi$ in the form

$$\Psi(x, y, t) = \Psi(y) \exp[i\alpha(x - ct)]$$  \hspace{1cm} (5)

Fig. 1. Neutral stability curves.

where \( \omega_c \) is a complex dimensionless frequency of perturbations, we can derive the generalized Orr–Sommerfeld equation for fluids with a rheological power law. For regions of flow \(-1 \leq y \leq 0\), in which \( dU/dy < 0 \), the generalized Orr–Sommerfeld equation has the form

\[
[(U - c)(D^2 - \alpha^2) - (DU)]\psi = \frac{(DU)^{n-2}}{aRe}[(D^3U - \alpha^2)]
\]

+ \((n - 1)(2n(DU)(DU)D^2 + [4\alpha^2(DU)^2 + n(DU)(DU)]
\]

+ \(n(n + 2)(D^2U)^2D^2 + 2(n - 2)\alpha^2(DU)(DU)D
\]

+ \(\alpha^2n[(DU)(DU) + (n - 2)(DU)^2)]\psi, \tag{6}\]

where \( D \equiv d/dy \), and \( Re = \rho U^2 y_c \frac{n-1}{\mu} \) is the generalized Reynolds number for power-law fluids. With \( n = 1 \), Eq. (6) changes into the Orr–Sommerfeld equation [3].

The boundary conditions for the function \( \psi \) are set at the half-width of the channel at the points \( y_1 = -1 \) and \( y_2 = 0 \), with the latter condition understood as the limit. For even perturbations, which are the most dangerous from the standpoint of flow stability, the boundary conditions are the following:

\[
\psi(y_1) = D\psi(y_1) = D\psi(y_2) = D^3\psi(y_2) = 0. \tag{7}\]

If \( \psi \) is given by the asymptotic expansion

\[
\psi(y) = \sum_{n=0}^\infty \frac{\psi^{(n)}(y)}{(\alpha Re)^n}, \tag{8}\]

the first pair of independent solutions of (6) is found from

\[
(U - c)(D^2 - \alpha^2) \psi - (DU)\psi = 0, \tag{9}\]

which is the equation of the zeroth approximation of \( \psi(y) \) in (8). The solutions of (9) can be found in the form of power series in \( y - y_c \), where \( y_c \) is the point at which \( U(y_c) = c \):

\[
\psi_1^{(0)} = (y - y_c) \sum_{k=0}^{\infty} a_k (y - y_c)^k,
\]

\[
\psi_2^{(0)} = \psi^{(0)} \ln (y - y_c) \frac{DU(y_c)}{DU(y_c)} + \sum_{k=0}^{\infty} b_k (y - y_c)^k, \tag{10}\]

where

\[
a_0 = b_0 = 1; a_1 = b_1 = \frac{1}{2ny_c}; a_2 = \frac{\alpha^2}{6} + \frac{1-n}{6n^2 y_c^2}; b_2 = \frac{\alpha^2}{2} + \frac{1+2n}{4n^2 y_c^2};
\]

\[
a_3 = \frac{\alpha^2}{18ny_c} + \frac{(1-n)(1-2n)}{24n^3 y_c^3}; b_3 = \frac{\alpha^2}{36ny_c} + \frac{4n^3-4n-3}{24n^3 y_c^3};
\]

\[
a_4 = \frac{\alpha^4}{120} + \frac{\alpha^2}{60n} \left( \frac{11}{12} - 1 \right) \frac{1}{y_c} + \frac{(1-n)(1-2n)(1-3n)}{120n^3 y_c^3};
\]

\[
b_4 = \frac{\alpha^4}{24} - \frac{\alpha^2(13 + 36n)}{432n^2 y_c^5} - \frac{6 - n - 20n^2 + 12n^3}{144n^3 y_c};
\]

\[
a_5 = \frac{23\alpha^4}{10800n y_c} - \frac{(180n^2 - 242n + 71)\alpha^2}{21600n^3 y_c^3} + \frac{(1-n)(1-2n)(1-3n)(1-4n)}{720n^3 y_c^3};
\]

\[
b_5 = \frac{\alpha^2(64 + 147n - 180n^2)}{3600n^2 y_c^5} + \frac{144n^4 - 300n^3 + 90n^2 + 80n - 29}{2880n^5 y_c};
\]

\[
a_6 = \frac{\alpha^6}{5040} - \frac{\alpha^3(270n - 233)}{453600n^3 y_c^7} - \frac{\alpha^2(780n^3 - 1270n^2 + 597n - 86)}{151200n^2 y_c^9} + \frac{(1-n)(1-2n)(1-3n)(1-4n)(1-5n)}{5040n^3 y_c};
\]