Variational formulation is given of the nonstationary heat conduction problem for two bodies; its numerical solution by the finite-element method (FEM) reduces to the solving of simultaneous linear algebraic equations.

1. Formulation of the Problem. A system of two bodies is given which are in contact with one another, each occupying the space \( V_1 \) and \( V_2 \) respectively with the corresponding boundaries \( B_1 \) and \( B_2 \). Let \( S \) be the common part of the boundary \( B_1 + B_2 \) (the surface of the contact) where one has a boundary condition of the third kind, \( \Gamma_1 \) and \( \Gamma_2 \) being the free parts of the boundary for which the heat flux is zero.

One then has the following system of equations:

\[
\begin{align*}
\frac{\partial \theta_1}{\partial t} &= \lambda_1 \frac{\partial^2 \theta_1}{\partial x^2} + \gamma_1 w_1 \quad (t > 0, \ x \in V_1), \\
\frac{\partial \theta_2}{\partial t} &= \lambda_2 \frac{\partial^2 \theta_2}{\partial x^2} \quad (t > 0, \ x \in V_2)
\end{align*}
\]

with the initial conditions

\[
\theta_1(x, 0) = \theta_1^0(x), \quad \theta_2(x, 0) = \theta_2^0(x)
\]

and the boundary conditions

\[
\begin{align*}
\lambda_1 \frac{\partial \theta_1}{\partial n} &= -\alpha (\theta_1 - \theta_1) \quad \text{on } S, \\
\lambda_2 \frac{\partial \theta_2}{\partial n} &= -\alpha (\theta_2 - \theta_1) \quad \text{on } S, \\
\lambda_1 \frac{\partial \theta_1}{\partial \tau} &= 0 \quad \text{on } \Gamma_1, \\
\lambda_2 \frac{\partial \theta_2}{\partial \tau} &= 0 \quad \text{on } \Gamma_2,
\end{align*}
\]

where \( B_1 = \Gamma_1 + S, \ B_2 = \Gamma_2 + S \). In the physical interpretation the subscript, or superscript, "1" corresponds to the metal, and the subscript "2" to the rollers.

2. Variational Formulation. Following [5] if the concept of convolution of two continuous functions \( f(x, t) \) and \( g(x, t) \) defined on \( V \times [0, \infty) \) is introduced by means of

\[
[f \ast g](x, t) = \int_0^t f(x, t-\tau) g(x, \tau) \, d\tau, \quad (x, \tau) \in V \times [0, \infty),
\]

where \( V \times [0, \infty) \) denotes the set which is the direct product of the region \( V \) and of the time interval \( [0, \infty) \) it can be seen that Eqs. (1) with the initial condition (2) are equivalent to the following relations:

\[
\begin{align*}
c_1 \frac{\partial \theta_1}{\partial t} &= \lambda_1 \frac{\partial^2 \theta_1}{\partial x^2} + c_1 \gamma_1 \theta_1 + \gamma_1 w_1 \quad \text{on } V_1 \times [0, \infty), \\
c_2 \frac{\partial \theta_2}{\partial t} &= \lambda_2 \frac{\partial^2 \theta_2}{\partial x^2} + c_2 \gamma_2 \theta_2 \quad \text{on } V_2 \times [0, \infty).
\end{align*}
\]

Indeed, by integrating, say, both sides of Eq. (1a) over \([0, t]\) and using (5) one obtains

\[
c_1 \int_0^t \frac{\partial \theta_1}{\partial t} (x, \tau) \, d\tau = \int_0^t \left( \lambda_1 \frac{\partial \theta_1}{\partial t} + \gamma_1 w_1 \right) \, d\tau
\]

or
\[ c_1 \gamma_1 \theta_1 (x, t) - c_1 \gamma_1 \theta_1 (x, 0) = \lambda_1 \theta_{1,tt} + \gamma_1 w_1. \]

The boundary conditions (3) and (4) can also be transformed in a similar manner resulting in equivalent relations:

\[ \lambda_1 \theta_{1,tt}^{(1)} = -\alpha \theta_1 (x, 0) \text{ on } S \times [0, \infty), \]
\[ \lambda_2 \theta_{2,tt}^{(2)} = -\alpha \theta_1 (x, 0) \text{ on } S \times [0, \infty), \]
\[ \lambda_1 \theta_{1,tt}^{(1)} = 0 \text{ on } \Gamma_1 \times [0, \infty), \]
\[ \lambda_2 \theta_{2,tt}^{(2)} = 0 \text{ on } \Gamma_2 \times [0, \infty). \]

To obtain the solution of the variational problem the concept of a feasible state \( R = \{ \theta_1, \theta_2 \} \) is introduced which represents the totality of two continuous functions \( \theta_1 (x, t) \) and \( \theta_2 (x, t) \) defined on \( V_1 \times [0, \infty) \) and \( V_2 \times [0, \infty) \). Then the solution of the problem (1)-(4) can be determined as a feasible state \( R = \{ \theta_1, \theta_2 \} \) such that it satisfies Eq. (1), the initial conditions (2) and the boundary conditions (3) and (4).

Further, let there be defined on the set of feasible states \( K \) for any \( t \in [0, \infty) \) the functional
\[ I (R) = \frac{1}{2} \int_{V_1} \left[ c_1 \gamma_1 \theta_1 \theta_1 + \lambda_1 \theta_{1,tt} \theta_1 + 2c_1 \gamma_1 \theta_1 w_1 - 2c_1 \gamma_1 \theta_1 \theta_1 \right] (x, t) dV_1 
+ \frac{1}{2} \int_{V_2} \left[ c_2 \gamma_2 \theta_2 \theta_2 + \lambda_2 \theta_{2,tt} \theta_2 + 2c_2 \gamma_2 \theta_2 w_2 - 2c_2 \gamma_2 \theta_2 \theta_2 \right] (x, t) dV_2 
+ \frac{1}{2} \int \left[ \alpha \theta_1 (0, t) \theta_1 \right] (x, t) dS + \frac{1}{2} \int \left[ \alpha \theta_2 (0, t) \theta_2 \right] (x, t) dS. \] (9)

Variational formulation is now given which is a generalization of the principles originally formulated for problems of linear elastodynamics [2, 3].

**THEOREM.** For a feasible state \( R = \{ \theta_1, \theta_2 \} \), \( R \in K \) to be a solution of the problem (1)-(4) it is necessary and sufficient that on \( K \) the condition be satisfied
\[ \delta I (R) = 0 \quad (0 \leq t < \infty). \] (10)

**Proof.** Having determined the variation of the functional (9) by taking into account the properties of the convolution [5] and the Gauss-Ostrogradskii theorem one obtains
\[ \delta I (R) = \int_{V_1} \left[ (c_1 \gamma_1 \theta_1 + \lambda_1 \theta_{1,tt}) \delta \theta_1 \right] (x, t) dV_1 
+ \int_{V_2} \left[ (c_2 \gamma_2 \theta_2 + \lambda_2 \theta_{2,tt}) \delta \theta_2 \right] (x, t) dV_2 
+ \int_{\Gamma_1} \left[ \lambda_1 \theta_{1,tt} \theta_{1} \right] (x, t) d\Gamma_1 
+ \int_{\Gamma_2} \left[ \lambda_2 \theta_{2,tt} \theta_{2} \right] (x, t) d\Gamma_2 
+ \int \left[ \alpha \theta_1 (0, t) \theta_1 \right] (x, t) dS + \int \left[ \alpha \theta_2 (0, t) \theta_2 \right] (x, t) dS. \] (11)

It is obvious that the variational formulation is necessary since by assuming \( R = \{ \theta_1, \theta_2 \} \) to be the solution of (1)-(4) together with (6)-(8) it can easily be seen that (10) follows from (11).

To prove the sufficiency two lemmas are formulated [3] similar to the fundamental lemma of the calculus of variations.

**LEMMA 1.** Let \( f \) be a continuous function \( V \times [0, \infty) \) and let us assume that
\[ \int_V [f \ast g] (x, t) dV = 0 \quad (0 \leq t < \infty) \]
for any function \( g \) which vanishes on \( B \times [0, \infty) \) (\( B \) is the boundary of the region \( V \)). Then \( f = 0 \) on \( V \times [0, \infty) \).

**LEMMA 2.** Let \( f \) be a piecewise-continuous function on \( B_1 \times [0, \infty) \) and let us assume that
\[ \int_{B_1} [f \ast g] (x, t) dB_1 = 0 \quad (0 \leq t < \infty). \]