ONE-DIMENSIONAL CONVECTIVE HEATING WITH A TIME-DEPENDENT HEAT-TRANSFER COEFFICIENT

Yu. S. Postol'nik

The problem of the symmetric convective heating of a plate, cylinder, and sphere with a time-dependent heat-transfer coefficient is solved by the method of averaging functional corrections.

The analytic study of the convective heating process with a time-varying heat-transfer coefficient requires the solution of the Fourier equation

\[ \frac{\partial T}{\partial t} = \frac{a}{x^m} \frac{\partial}{\partial x} \left[ x^m \frac{\partial T}{\partial x} \right], \tag{1} \]

describing the symmetric heating of a plate \((m = 0)\), cylinder \((m = 1)\), or sphere \((m = 2)\) with a boundary condition of the third kind corresponding to Newton's law

\[ \lambda \frac{\partial T}{\partial x} \bigg|_{x = R} = \alpha(t) [T_a - T_s(t)]. \tag{2} \]

It is assumed that the coefficients \(a\) and \(\lambda\), the ambient temperature \(T_a\), and the initial temperature \(T_0\) are constants:

\[ T(x, 0) = T_0 = \text{const.} \tag{3} \]

We shall stipulate a second boundary condition later. If we introduce the dimensionless variables

\[ \xi = \frac{x}{R}; \quad \tau = \frac{at}{R^2} = Fo; \quad Bi(\tau) = \frac{\alpha(\tau) R}{\lambda}; \]

\[ \theta(\xi, \tau) = \frac{T(x, t)}{T_a} \tag{4} \]

and, following [1], the new function

\[ u(\xi, \tau) = \ln \left[ 1 - \theta(\xi, \tau) \right]. \tag{5} \]

Eqs. (1)-(3) are transformed to

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + \frac{m}{\xi} \frac{\partial u}{\partial \xi} + \left( \frac{\partial u}{\partial \xi} \right)^2; \tag{6} \]

\[ \frac{\partial u}{\partial \xi} \bigg|_{\xi = 1} = -Bi(\tau); \tag{7} \]

\[ u(\xi, 0) = \ln (1 - \theta_0) = u_0. \tag{8} \]

The term \((\partial u/\partial \xi)^2\) of Eq. (6) is omitted in [1], thus restricting the discussion to thin bodies.

In accord with the conventional [2] engineering model of the heating process we consider two successive stages: inertial (warming up of the body) and regular (heating over the whole cross section). The problem is solved by the method of averaging functional corrections [3-5].

Inertial Heating \( (0 \leq \tau \leq \tau_0) \). We assume that the temperature distributions at the boundary between the heated and unheated zones are joined. In this case there must be added to Eq. (6) with initial condition (8) and boundary condition (7) the joining conditions

\[
\begin{align*}
& u(\xi, \tau)|_{\xi=\rho(t)} = u_0, \\
& \frac{\partial u}{\partial \xi} |_{\xi=\rho(t)} = 0;
\end{align*}
\]

where

\[
\rho(t) = \frac{r(t)}{R};
\]

\( r(t) \) is the width of the unperturbed zone of the cross section of the body, i.e., the distance from the center of the cross section to the front of the moving thermal perturbation.

As in [4, 5] we set

\[
\frac{\partial^2 u_\xi}{\partial \xi^2} = f_\xi (t),
\]

where

\[
f_\xi (t) = \frac{1}{1 - \rho(t)} \int_{\rho(t)}^{1} \left[ \frac{\partial u}{\partial \tau} - \left( \frac{\partial u}{\partial \xi} \right)^2 - m \frac{\partial u}{\partial \xi} \right] d\xi.
\]

Integrating (12) twice with respect to \( \xi \) and using (9) and (10) we have

\[
u_\xi (\xi, \tau) = u_0 - \frac{\text{Bi}(t)}{2[1 - \rho(t)]} \left[ \xi - \rho(t) \right]^2, \quad (\rho < \xi < 1).
\]

From (7) we find

\[
f_\xi (t) = \frac{\text{Bi}(t)}{1 - \rho(t)}.
\]

Substituting (14) and (15) into (13) and making a number of transformations we find the following differential equation for the remaining unknown function \( \rho(\tau) \):

\[
\frac{d}{d\tau} \left[ \text{Bi}(\tau) \left[ 1 - \rho(\tau) \right]^2 \right] + 2 \text{Bi}^2(\tau) \left[ 1 - \rho(\tau) \right] = 6 (m + 1) \text{Bi}(\tau).
\]

Here by analogy with [5] we have omitted the terms containing the factor \( \rho(\tau) \ln \rho(\tau) \).

Introducing the notation

\[
\begin{align*}
g(\tau) &= \gamma \text{Bi}(\tau) \beta(\tau), \\
\beta(\tau) &= 1 - \rho(\tau),
\end{align*}
\]

we rewrite Eq. (16) in the form

\[
g(\tau) \frac{dg}{d\tau} + \text{Bi}(\tau) \sqrt{\text{Bi}(\tau)} g(\tau) = 3 (m + 1) \text{Bi}(\tau).
\]

Equation (19) is a special case of the well-known Abel's equation of the second kind [6].

Setting

\[
\begin{align*}
g(\tau) &= v(\tau) + \Phi(\tau), \\
\Phi(\tau) &= - \int [\text{Bi}^{3/2}(\tau) d\tau,
\end{align*}
\]

234