DISCONTINUOUS SOLUTIONS IN PROBLEMS OF NONLINEAR HEAT CONDUCTION

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We propose a method of constructing discontinuous solutions for nonlinear problems in the theory of heat conduction. The method is a modification of the pivot methods for the solution of linear boundary-value problems. As an illustration we present numerical solutions of two problems.

1. Let us examine the following linear boundary-value problem:

\[ c_1 y'' + c_2 y' + c_3 y + c_4 = 0, c_1 = c_1(x), \quad x \in (a, b); \]
\[ (a_0 y + b_0 y')|_{x=a} = y_0, \quad (a_0 y + b_0 y')|_{x=b} = y_1. \]

The solution of this problem can be constructed by the pivot method [1, 2]. Having approximated (1) by a system of finite-difference relations

\[ \frac{c_1^k y_{k+1} - 2y_k + y_{k-1}}{h^2} + \frac{c_2^k y_{k+1} - y_k}{2h} + c_3^k y_k + c_4^k = 0, \]

where \(c_1^k = c_1(x_k), \quad y_k = y(x_k), \quad x_k = a + kh, \quad 1 \leq k \leq n - 1,\) we can derive the formulas for the pivot coefficients

\[ a_1^k \left[ 2c_1^k - a_1^{k-1} \left( c_1^k - \frac{h}{2} c_2^k \right) - h^2 c_3^k \right] = c_4^k + \frac{h}{2} c_2^k, \]
\[ a_2^k \left[ 2c_1^k - a_1^{k-1} \left( c_1^k - \frac{h}{2} c_2^k \right) - h^2 c_3^k \right] = a_2^{k-1} \left( c_1^k - \frac{h}{2} c_2^k \right) + h^2 c_4^k. \]

The formula for the reverse pivot

\[ y_k = a_1^k y_{k+1} + a_2^k \]

with the coefficients \(a_1^k\) and \(a_2^k\), which can be calculated according to (4), yields a solution for the boundary-value problem (1) and (2) that is correct to \(o(h)\).

In solving the boundary-value problems by the pivot method (3)-(5) the following conditions are significant: first of all, the solution of (1) is assumed to be continuous-differentiable on \((a, b)\); secondly, the calculation of the pivot coefficient in accordance with the recurrent formulas (4) or the analogous formulas (see [2]) assumes knowledge of the first coefficients \(a_1^1\) and \(a_2^1\) which are found from the left-hand boundary condition. In the general case this can be done only when at least one of the boundary conditions is linear.

In section 2 we construct a modification of the pivot method by means of which it becomes possible to remove these limitations.

2. We have to find a solution for (1) to satisfy boundary conditions of the general form

\[ \varphi_a(y_a, y_a', y_n, y_n') = 0, \quad \varphi_b(y_b, y_b', y_n, y_n') = 0. \]

Moreover, with a finite number of points \(\xi_i \in \mathbb{R} (a, b)\) the solutions and its derivative may experience discontinuities which are described by the following linear conditions:

\[ a_1^p y_k + a_2^p y_k += b_1^p y_k + b_2^p y_k + c_3^p, \]
\[ a_1^p y_k + a_2^p y_k += b_1^p y_k + b_2^p y_k + c_3^p, \quad p = 1, 2, 3, \ldots, m, \]
where \( \dot{y}_k^\pm = y'(\xi_k \pm 0) \), \( \ddot{y}_k^\pm = y''(\xi_k \pm 0) \).

We note that at the point \( \xi_p \) Eq. (1) loses significance and, of course, cannot be satisfied; the coefficients \( c_i(x) \) at the points may therefore also experience discontinuities. Replacing \( \dot{y}_k \) by the finite-difference expressions
\[
\dot{y}_k^- = \frac{y_{k-2}^- - 4y_{k-1}^- + 3y_k^-}{2h} + o(h),
\]
and using formulas (5) of the reverse pivot for the point \( x^k = \xi_p + h \), we can bring (7) to the form
\[
A_{12}^k y_k^- + 2A_{12}^k y_k^- = B_{12}^k y_{k+2}^- + D_{12}^k,
\]
\[
A_{22}^k y_k^- + A_{22}^k y_k^- = B_{22}^k y_{k+2}^- + D_{22}^k.
\]
It can be demonstrated that in all cases of physical significance the determinant of this system does not vanish identically with respect to the step into which the segments \([\xi_p, \xi_{p+1}]\) of the interval \((a, b)\) have been separated. With solution of the system we derive the formulas of reverse pivot for the point \( x_k = \xi_p \)
\[
y_k^- = a_{1p}^- y_{k+2}^- + a_{2p}^- \quad p = 1, 2, \ldots, m.
\]
Unlike (4), the coefficients \( a_{1p}^\pm, a_{2p}^\pm \) are functions not only of the pivot coefficients that have been calculated, but also of \( a_{1p+1}^k, a_{2p+1}^k \). The latter are found from (8), (5), and (4), as well as from Eq. (3) for \( x_k = \xi_p + \delta \).

The recurrence formulas derived in this manner for the pivot coefficients, in conjunction with (8), make it possible to pass through the points \( \xi_p \) of the solution discontinuities described by (7) without resorting to iterations.

Let us now turn to the boundary conditions (6). Having introduced the notations \( y_0 = \alpha, y_n = \beta \) and assuming \( \alpha \) to be known, we find the expressions for the first pivot coefficients:
\[
a_1^0 = 1 + \frac{h^2}{2} \frac{c_1}{c_1 + hc_2}, \quad a_2^0 = \frac{h}{2} \frac{hc_1 - 2\alpha \left( c_1 + \frac{h}{2} c_2 \right)}{c_1 + hc_2}.
\]

Having substituted these expressions into (4), we note that all \( a_1^k \) are not functions of \( \alpha \), while \( a_2^k \) are linear functions of
\[
a_2^k = y^k a + \delta^k, \quad x_k = \xi_p.
\]
It is easy to demonstrate that these properties are also present in the coefficients \( a_{1p}^\pm, a_{2p}^\pm \).

Fig. 1. The functional of integral nonadjustment as a function of the parameter \( \delta \) and the iteration number.
Fig. 2. Integral curves \( y^k(x) \) as a function of the iteration number.