TRANSIENT HEAT CONDUCTION IN HOLLOW SPHERES WITH A MOVING INNER BOUNDARY

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The finite integral transform method is used to obtain the solution of unsteady heat conduction problems for a hollow sphere with a moving internal boundary and various boundary conditions at the outer surface. For the solution of the problems of interest integral transform formulas are presented with kernels (16), (20), and (24) and the corresponding inversion formulas (18), (22), (26), (29) and characteristic equations (17), (21), (25), (28), (31), (33).

Using a method analogous to that used in [1, 2], we shall obtain solutions to spherically-symmetric problems involving a moving inner boundary and various outer boundary conditions.

Up to the moment when the boundary begins to move, the mathematical formulation of the problem is

\[
\frac{\partial t}{\partial \tau} = a \left( \frac{\partial^2 t}{\partial r^2} + \frac{2}{r} \frac{\partial t}{\partial r} \right), \quad R_1 < r < R_2, \quad \tau > 0,
\]

\[
t(r, \tau) \big|_{\tau=0} = 0,
\]

\[
\frac{\partial t(r, \tau)}{\partial r} \bigg|_{r=R_1} = -\frac{q_1(\tau)}{\lambda},
\]

\[
a(\tau) \frac{\partial t}{\partial r} \bigg|_{r=R_2} \left[ \beta(t(r, \tau)) \right]_{r=R_2} + \gamma(\tau) = 0.
\]

Using the appropriate integral transform, one can obtain the solution to (1)-(4). Assume this solution to be \( t = \Theta = r^{\frac{1}{2}} f_0(r, \tau) \), where

\[
\Theta = r^{\mu} t(r, \tau).
\]

From the condition

\[
t(\tau_0) = f_0(r, \tau_0)
\]

one can find \( \tau_0 = \Phi(R_1), \) i.e., the time at which the boundary begins to move. From the time \( \tau = \tau_0 \) on, the boundary \( r = R_1 \) moves according to the law \( r = s(\tau) \). In that case the mathematical formulation will differ from (1)-(4), first because of the conditions

\[
t(r, \tau) \big|_{\tau=\tau_0} = f_0(r, \tau_0),
\]

\[
t(r, \tau) \big|_{r=s(\tau)} = t(\tau)
\]

and, second, because of the additional heat-balance condition at the moving boundary

\[
\lambda \frac{\partial t(r, \tau)}{\partial r} \bigg|_{r=s(\tau)} = -\frac{Q_0(\tau)}{4\pi r^2} + \rho F \frac{ds}{d\tau}.
\]

Now it is required to find the temperature field for \( \tau > \tau_0, s(\tau) \leq r \leq R_2 \) and the law of motion of the boundary. We shall divide the arbitrary time interval \( T = T_0 - \tau_0 \) into \( n \) parts, as in [1, 2], corresponding to the times \( \tau_1, \tau_2, \ldots, \tau_n = T_0, \Delta \tau_1 = \tau_{i+1} - \tau_i \). The time \( \tau_i \) corresponds to the point \( O_i = (\tau_i, s(\tau_i)) \) on the \( r \) axis, and we shall assume that the point \( O_i \) is stationary for \( \tau_i < \tau < \tau_{i+1} \) and jumps instantaneously to \( O_{i+1} = (\tau = \tau_{i+1} \) at \( \tau = \tau_{i+1} \). As a result we obtain a step-like \( s_i(\tau) \) instead of \( s(\tau) \). It can be proved easily that the function \( t_i(t_i, \tau) \) satisfies (1) with \( r > \tau_i, \tau_i > \tau_i \), the initial condition \( t_i(t_i, \tau) = f_i(t_i, \tau_i) = t_{i-1}(t_i, \tau_i) \), boundary condition (6) at \( \tau = s_i = s(\tau_i) \), and boundary condition (9).

Using (6), one can solve the above problem by the method of finite integral transforms. Assuming the solution to be

\[
f_i(t_i, \tau) = f_i(t_i, \tau_i)
\]
and taking into account that
\[ f_i(r, \tau_i) = t_{i-1}(r, \tau_i), \]
we can express \( t_{i-1} \) in (10) in terms of \( t_{i-2} \), then express \( t_{i-2} \) in terms of \( t_{i-3} \), etc. The result is
\[ t_i(r, \tau) = F_i \left( f_0(r, \tau_0), r, R_2, r_1, r_{i-1}, \ldots, R_1, \tau, \tau_i, \ldots, \tau_0 \right). \]
(12)

Using (12), we can determine the unknown values \( \tau_1 = s(\tau_1) \), i.e., the approximate law of motion of the boundary. Rewrite Eq. (9) in the form
\[ \frac{ds}{d\tau} = \frac{\lambda}{\rho F} \left. \frac{\partial t_i(r, \tau)}{\partial r} \right|_{r=r_i} + \frac{Q_0(\tau)}{4\pi \rho F s_i^2}. \]
(13)

The right-hand side of (13) is a known function of \( \tau \). Integrating (13) over \( \tau \) from \( \tau_i \) to \( \tau_{i+1} \) and adding the equations for \( i = 0, 1, 2, \ldots, l \) (\( l \leq n \)), we obtain
\[ s(\tau_{i+1}) - s(\tau_i) = \sum_{i=0}^{l-1} \left[ s(\tau_{i+1}) - s(\tau_i) \right] = \sum_{i=0}^{l-1} \int_{\tau_i}^{\tau_{i+1}} \left[ \frac{\lambda}{\rho F} \left. \frac{\partial t_i(r, \tau)}{\partial r} \right|_{r=r_i} + \frac{Q_0(\tau)}{4\pi \rho F s_i^2} \right] d\tau. \]
(14)

Knowing \( s(\tau) \) from (12), we obtain the functions \( t_i(r, \tau) \). We shall now illustrate this method by means of specific examples.

Hollow sphere with boundary condition of the first kind at the outer surface \( (\alpha(\tau) = \beta(\tau) = 1, \gamma(\tau) = \varphi_2(\tau)) \).

Applying to the function \( \Theta(t, \tau) \) the integral transform
\[ \overline{\Theta}_{\nu_n}(\tau) = \int_{R_1}^{R_2} r \Theta(r, \tau) W_0 \left( \frac{\nu_n}{R_1} r \right) dr, \]
(15)

with the kernel
\[ W_0 \left( \frac{\nu_n}{R_1} r \right) = \frac{2}{\pi \nu_n k_{1/2}} \left( \frac{R_1}{r} \right)^{1/2} \sin \nu_n \left( k - \frac{r}{R_1} \right), \]
(16)

where \( \nu_n \) are the roots of the characteristic equation
\[ \tan \nu_n (k - 1) = -\nu_n \]
(17)

with \( k = R_2/R_1 \), and using the inversion formula
\[ W_0^{-1} \left[ \Theta_{\nu_n}(\tau) \right] = \Theta(t, \tau) = r^{1/2} t(r, \tau) = \frac{\pi^2 k}{R_1} \sum_{n=1}^{\infty} \frac{\nu_n^3}{2 \nu_n (k - 1) - \sin 2 \nu_n (k - 1)} W_0 \left( \frac{\nu_n}{R_1} r \right), \]
(18)

we obtain the solution
\[ \Theta(t, \tau) = r^{1/2} t(r, \tau) = \frac{\pi^2 k}{\alpha R_1^{1/2}} \sum_{n=1}^{\infty} \frac{\nu_n^3}{2 \nu_n (k - 1) - \sin 2 \nu_n (k - 1)} \times \]
\[ \times \int_0^\tau \left[ \frac{R_1}{\lambda} q_1(\theta) W_0(\nu_n) - \nu_n k^{1/2} \varphi_2(\theta) W_0(\nu_n k) \right] \times \]
\[ \times \exp \left[ -\frac{\nu_n}{\alpha}(\beta_1, \tau) - \beta_0(1, \theta) \right] d\theta W_0 \left( \frac{\nu_n}{R_1} \right), \]
(19)