OPTIMUM SHAPE FOR SINGLE EMISSION ELEMENTS

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Optimum laws of variation in cross-sectional areas cooled by emission from solid and hollow radiation elements of various shapes are found.

1. Let us consider the problem of designing an optimum radiation-cooled heat-conduction pin of minimum weight whose area and cross-sectional perimeter would be defined by the equalities (Fig. 1a):

\[ F = k_1 y^2; \quad \Pi = k_2 y. \]  

We will consider rather long pins for which the equation of heat transfer along the pin and the law governing thermal radiation are valid in the following form:

\[ k_1 y^2 \lambda \frac{dT}{dx} = -Q, \]  
\[ k_2 y \sigma T^4 dx = -dQ. \]  

The optimum relationship \( y(x) \) must ensure minimum pin volume

\[ V = \int_0^{x_L} k_1 y^2 dx \]  

for given initial heat flux \( Q_0 \) and temperature \( T_0 \) (Fig. 1a).

We introduce the nondimensional variables

\[ \bar{Q} = \frac{Q}{Q_0}; \quad \bar{T} = \frac{T}{T_0}; \quad \bar{x} = x / \left( \frac{Q_0 k_1}{\sigma k_2^2 \lambda T_0^4} \right)^{1/3}; \]  
\[ \bar{y} = y / \left( \frac{Q_0^2}{\sigma k_1^2 \lambda T_0^6} \right)^{1/3}; \quad \bar{V} = V / \left( \frac{Q_0 k_1^3}{\lambda \sigma k_2^4 \lambda T_0^7} \right)^{1/3}. \]  

In these variables Eqs. (2)-(4) are written in the form:

\[ \frac{\bar{y}^2}{\bar{T}} \frac{d\bar{T}}{d\bar{x}} = -\bar{Q}; \]  
\[ \bar{y} \bar{T}^4 d\bar{x} = -d\bar{Q}; \]  
\[ \bar{V} = \int_0^{x_L} \frac{\bar{y}^2}{\bar{T}} d\bar{x}. \]  

At the end of the pin when \( \bar{x} = \bar{x}_L \), we know only the value of the variable \( \bar{Q} = 0 \), and it is therefore advisable to express all of the variables as a function of \( \bar{Q} \). Expression (8) is then rewritten as

\[ \bar{V} = -\int_0^{x_L} \left( \frac{\bar{Q}}{\bar{T}^{16} \frac{d\bar{T}}{d\bar{Q}}} \right)^{1/3} d\bar{Q} \]  

and the formulated problem will correspond to the sought minimum of the functional (9) whose Euler equation will have the form

\[ \frac{d^2 \bar{T}}{d\bar{Q}^2} + \frac{16}{\bar{T}} \left( \frac{d\bar{T}}{d\bar{Q}} \right)^3 - \frac{1}{4 \bar{T}} \frac{d\bar{T}}{d\bar{Q}} = 0. \]  

The general solution of this equation is written in the form

\[ \bar{T} = C_1 \left( \frac{5}{5} \bar{Q}^{5/3} + C_2 \right)^{1/5}. \]  

For the optimum pin contour it follows from the natural limit boundary condition at the right-hand end that \( C_2 = 0 \). At the base of the pin we must have \( \bar{Q} = 1 \) and \( \bar{T} = 1 \), and therefore \( C_1 = 1 \).

Thus considering (6)-(8), we have the following relationships characterizing the optimum pin:

\[ \bar{T} = \bar{Q}^{5/8}; \quad \bar{x} = 2.58 \left( 1 - \bar{Q}^{35/36} \right); \]  
\[ \bar{T} = \left( 1 - \frac{\bar{x}}{2.58} \right)^{1/30} \]  

for given initial heat flux \( Q_0 \) and temperature \( T_0 \) (Fig. 1a).
The various transverse cross-sectional shapes will be defined by the coefficients $k_1$ and $k_2$. For example, having substituted $k_1 = \pi$ and $k_2 = 2(1 + \xi)$, we will have a pin with a rectangular cross section and a side ratio equal to $\xi$ (Fig. 1c), etc.

We note that all of the results obtained above are also valid for hollow pins (Fig. 1d) in which there is no radiative heat exchange between the inside surfaces, and the wall thickness $\delta$ changes according to a definite law along the pin. For example, for a circular pin (Fig. 1f) the ratio $(y - \delta)/y$ must be constant. Here

$$k_1 = \pi \left[1 - \left(\frac{y - \delta}{y}\right)^2\right]; \quad k_2 = \pi.$$

For a pin with a rectangular profile whose sides are $a$ and $b$ long (Fig. 1g) it is necessary that

$$\frac{1 - 2\delta}{a} \left(\frac{\xi - 2\delta}{a}\right) = \text{const.}$$

Here

$$k_1 = \left[\xi - \left(1 - \frac{2\delta}{a}\right)\left(\frac{\xi - 2\delta}{a}\right)\right]; \quad k_2 = 2(1 + \xi).$$

We also note that the effectiveness of the optimum pin is independent of the shape of its lateral cross section

$$\Theta = \frac{Q_0}{G_p} \int_0^{x_L} k_2 y \sigma T_0^4 dx = 0.706. \quad (13)$$

However, the ratio of the removed flux to the weight of the pin is a strong function of the cross-sectional shape of the pin

$$\frac{Q_0}{G_p} = \frac{1}{1.91 \cdot \Psi} \left(\frac{\lambda k_1^4 \gamma T_0^4}{Q_0^4}\right)^{\frac{1}{3}} \left(\frac{K_1}{K_2}\right)^{\frac{1}{3}}. \quad (14)$$

Relationship (14) shows in particular that when the values of $Q_0$, $\lambda$, and $T_0$ are fixed, a solid pin with a lateral cross section in the form of a circle exhibits the smallest value for $Q_0/G_p$.

The results obtained above pertain to extremely pointed pins (when $x = x_L$, $y = 0$ and the first three derivatives of $y$ with respect to $x$ are equal to zero), as well as to the zero temperature $T_L$. Let us examine the class of optimum pins for $T_L \neq 0$. Here the function $\Theta(Q)$ is written as

$$\Phi = \left[(1 - T_L)Q + T_L\right]^\frac{1}{3}, \quad (15)$$

while the contour shape is defined by the following system of equations:

$$\frac{-3}{(68/5)Q} \left[\left(1 - T_L\right)Q + T_L\right]^{\frac{1}{3}} = \frac{(68/5)Q^2 \left[(1 - T_L)Q + T_L\right]^{\frac{1}{3}}}{1 - T_L}, \quad (16)$$

$$\tilde{x} = \frac{(68/5)\gamma}{3} \times \frac{\frac{1}{Q} \int_0^Q \left[(1 - T_L)Q + T_L\right]^{\frac{1}{3}} dQ}{1 - T_L}, \quad (17)$$

The effect of $T_L$ on the volume of the optimum pin is

$$V = \int_0^{x_L} y^{-2} dx = \frac{V_{opt}}{(1 - T_L)}, \quad (18)$$

The results obtained in calculating the optimum contours for pins with various values of $T_L$ are shown in Fig. 2 (Sidorenko did the calculations). We note that when $T_L = 0$ near the end of the pin, the derived solutions must naturally be refined, since the condition of a flattened contour is not satisfied there.

Let us compare the considered optimum pins with conical pins (the dashed lines in Fig. 1a). The area and perimeter of the lateral cross section in this case will be determined by Eqs. (1) in which $y$ is defined in terms of $x$:

$$y = (L - x) \tan \frac{\alpha}{2}. \quad (19)$$

With consideration of (19), Eqs. (2) and (3) lead to the differential equation

$$(\bar{L} - \bar{x}) \frac{dT}{dx} = \frac{2}{N} \frac{dT}{dx} - N \bar{T}^4 = 0, \quad (20)$$

where

$$\bar{x} = \frac{x}{x_L}; \quad \bar{L} = \frac{x_L}{x_L}; \quad N = \frac{\lambda k_1 \tan \alpha}{\frac{a}{2}}, \quad (21)$$

($x_L$ is the length of the truncated cone, see Fig. 1a). The boundary conditions for Eq. (20) may be assumed to be

$$\bar{T} = 1 \text{ when } \bar{x} = 0; \quad \frac{d\bar{T}}{dx} = 0 \text{ as } \bar{L} \to 1. \quad (22)$$

A solution for (20) was derived numerically on a computer. The effectiveness of the conical pin as a function of $N$

$$\Theta_c = \frac{Q_0}{G_p} \left[\int_0^{x_L} k_2 (L - x) \tan \frac{\alpha}{2} \sigma T_0^4 dx\right]^{-1}, \quad (23)$$

is shown in Fig. 3 (Potapov did the calculations). These relationships are valid for conical pins having any lateral cross-sectional shape (if there is no self-