SOLUTION OF A SYSTEM OF HEAT AND MASS TRANSFER EQUATIONS BY THE
METHOD OF STRAIGHT LINES

A. S. Bomko and V. I. Zhidko


UDC 536.2.01

An examination is made of the method of straight lines for solution of
differential equations in partial derivatives relating to a system of heat
and mass transfer equations.

In investigation of kinetics of a drying process there is a need to solve a linear system of heat and mass transfer equations [1]. The network method [2],
for example, is applied with success to this problem and among other methods of solving equations in par-
tial derivatives, the method of straight lines [3] is
well known. A number of problems in heat transfer,
mechanics, and hydrodynamics have been solved by
this method [4–6]. The merit of the method is the
fact that solution of the problem is reduced, eventual-
ly, to be a standard system of ordinary differential
equations. For this reason, the method has an
advantage compared with the network method when
analog computers are used. Moreover, for linear sys-
tems, the method allows us to obtain analytical de-
pendences of the desired functions on one of the in-
dependent variables, e.g., the Fourier number, for
definite values of the other variable. It should be
noted, however, that in the majority of cases of prac-
tical importance, a solution cannot be obtained in
explicit form, and the method of straight lines en-
counters the same difficulty as the network method.

The essence of the straight lines method and some
of its applications are described in detail in refer-
ences [7–9, 13, 14].

We shall examine the linear system to which we
may reduce the problem of heating and dehydration
of a capillary-porous sphere in an external medium
of constant temperature and with infinitely large veloc-
ity of propagation of heat and moisture:

\[
\begin{align*}
\frac{\partial t}{\partial F_0} &= A_{11} \Delta t + A_{12} \Delta u, \\
\frac{\partial u}{\partial F_0} &= A_{21} \Delta t + A_{22} \Delta u,
\end{align*}
\]

where

\[
\begin{align*}
A_{11} &= 1 + \varepsilon K_0 L_0 P_n, \\
A_{12} &= \varepsilon K_0 L_0, \\
A_{21} &= L_0 P_n, \\
A_{22} &= L_u,
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
t(r, 0) &= t_0, \\
u(r, 0) &= u_0
\end{align*}
\]

and boundary conditions of the 3rd kind

\[
\begin{align*}
-\frac{\partial t}{\partial r} (1, F_0) &= B_i_r [1 - t (1, F_0)] - (1 - \varepsilon) K_0 L_0 B_i_m [u (1, F_0) - u_0] = 0, \\
\frac{\partial u}{\partial r} (1, F_0) + P_n \frac{\partial t}{\partial r} (1, F_0) + B_i_m [u (1, F_0) - u_0] &= 0.
\end{align*}
\]

The dimensionless numbers are as follows:

\[
L_u = a_m / a_t, \quad B_i_r = \alpha R / \lambda, \quad B_i_m = \beta R / a_m, \\
K_0 = \rho u_0 T_0 / c, \quad P_n = \delta T_0 / a, \\
F_0 = a_t \varepsilon R^2.
\]

A condition of finiteness for the functions \( t \) and \( u \) is imposed, naturally, at the center of the sphere.

In accordance with the method we shall replace the
derivatives with respect to \( r \) in (1) by the finite dif-
ference relations:

\[
\begin{align*}
\frac{\partial t}{\partial F_0} &\approx \Delta t \approx \frac{t_{k+1} - 2 t_k + t_{k-1}}{h^2}, \\
\frac{\partial u}{\partial F_0} &\approx \Delta u \approx \frac{t_{k+1} - 2 t_k + t_{k-1}}{h^2},
\end{align*}
\]

Taking account of the fact that \( r_k = k h \) and

\[
\begin{align*}
(\Delta t)_k &= \frac{\partial t}{\partial r} \approx \frac{2}{h} \frac{\partial t}{\partial r}, \\
(\Delta u)_k &= \frac{\partial u}{\partial r} \approx \frac{2}{h} \frac{\partial u}{\partial r},
\end{align*}
\]

we obtain

\[
\begin{align*}
(\Delta t)_k &\approx \frac{1}{kh^2} [2k^2 h (k + 1) - 2k(k - 1) t_{k-1}], \\
(\Delta u)_k &\approx \frac{1}{kh^2} [2k^2 h (k + 1) u_{k+1} - 2ku_k + (k - 1) u_{k-1}],
\end{align*}
\]

where \( h \) is the step size with regard to \( r \) (dimen-
sionless); \( k = 1, 2, 3, \ldots, n; \) the number of internal
straight lines is equal to \( n. \)

At the boundary (straight line with number \( k = n + 1 \)) we have:

\[
\begin{align*}
(\Delta t)_{k+1} &= \frac{\partial t}{\partial r} \approx \frac{2}{r_{n+1}} \frac{\partial t}{\partial r}, \\
(\Delta u)_{k+1} &= \frac{\partial u}{\partial r} \approx \frac{2}{r_{n+1}} \frac{\partial u}{\partial r},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial t}{\partial r} &= t_{n+1} - h \frac{\partial t}{\partial r} + \frac{h^2}{2} \frac{\partial^2 t}{\partial r^2} - \cdots, \\
\frac{\partial u}{\partial r} &= u_{n+1} - h \frac{\partial u}{\partial r} + \frac{h^2}{2} \frac{\partial^2 u}{\partial r^2} - \cdots,
\end{align*}
\]

\[
\begin{align*}
(\Delta t)_{k+1} &\approx \frac{2}{h} \left( \frac{\partial t}{\partial r} - \frac{t_{n+1} - t_n}{h} \right) + \frac{2}{r_{n+1}} \frac{\partial t}{\partial r}, \\
(\Delta u)_{k+1} &\approx \frac{2}{h} \left( \frac{\partial u}{\partial r} - \frac{u_{n+1} - u_n}{h} \right) + \frac{2}{r_{n+1}} \frac{\partial u}{\partial r}.
\end{align*}
\]
We write down \((\Delta u)_n\) at the boundary in similar fashion.

Allowing for the fact that \(r_n+1 = 1\), we obtain, for three straight lines, for example, the following system of ordinary differential equations:

\[

t_i = h^{-2} [A_{11} (2t_i - 2t_j) + A_{12} (2u_i - 2u_j)],
\]

\[
u_i = h^{-2} [A_{11} (2t_i - 2t_j) + A_{12} (2u_i - 2u_j)],
\]

\[

t_2 = h^{-1} \left[ A_{11} \left( \frac{3}{2} t_3 - 2 t_2 + \frac{1}{2} t_1 \right) + + A_{12} \left( \frac{3}{2} u_3 - 2 u_2 + \frac{1}{2} u_1 \right) \right],
\]

\[
u_2 = h^{-1} \left[ A_{11} \left( \frac{3}{2} t_3 - 2 t_2 + \frac{1}{2} t_1 \right) + + A_{12} \left( \frac{3}{2} u_3 - 2 u_2 + \frac{1}{2} u_1 \right) \right],
\]

\[

t_3 = h^{-1} \left[ A_{11} \left( \frac{4}{3} t_4 - 2 t_3 + \frac{2}{3} t_2 \right) + + A_{12} \left( \frac{4}{3} u_4 - 2 u_3 + \frac{2}{3} u_2 \right) \right],
\]

\[
u_3 = h^{-1} \left[ A_{11} \left( \frac{4}{3} t_4 - 2 t_3 + \frac{2}{3} t_2 \right) + + A_{12} \left( \frac{4}{3} u_4 - 2 u_3 + \frac{2}{3} u_2 \right) \right],
\]

\[

t_i = A_{11} (\Delta t_i) + A_{12} (\Delta u_i),
\]

\[
u_i = A_{11} (\Delta t_i) + A_{12} (\Delta u_i),
\]

where

\[
(\Delta t_i) = 2 \frac{\partial t}{\partial r} (1, Fo) + 2 \frac{h}{t} \left[ \frac{\partial t}{\partial r} (1, Fo) - \frac{t_i - t_j}{h} \right],
\]

\[
(\Delta u_i) = 2 \frac{\partial u}{\partial r} (1, Fo) + 2 \frac{h}{u} \left[ \frac{\partial u}{\partial r} (1, Fo) - \frac{u_i - u_j}{h} \right].
\]

The initial conditions remain as before, while the derivatives with respect to functions \(t\) and \(u\) on the sphere surface are substituted from the boundary conditions (3) into the system (5). Thus the boundary conditions are included in the system in the form of differential equations. However, the order of the system may be lowered by two, if the boundary conditions are considered in the form of algebraic equations, by replacing derivatives of the first order with respect to \(r\) by finite difference relations. This may prove to be important when working with a small number of straight lines.

The convergence of the method of straight lines for linear systems, and also for systems with variable coefficients in a rectangular region follows from the lemma of section 3 of [10], under the following sufficient condition:

\[
\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial u} < \text{const} < 0,
\]

\[
\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial u} \leq \text{const} < 0,
\]

where \(\varphi\) and \(\psi\) are the right sides in the boundary condition of the 3rd kind:

\[
\frac{\partial t}{\partial r} (R, \tau) = \varphi [t, t (R, \tau), u (R, \tau)],
\]

\[
\frac{\partial u}{\partial r} (R, \tau) = \psi [t, t (R, \tau), u (R, \tau)].
\]

Here we assume the existence and uniqueness of the solution of the mixed problem (1)–(3), and the continuity and sufficient smoothness of the coefficients and of the solutions themselves.

In the case of the linear system (1) the condition of convergence has the form

\[
Bi_v > (1 - \varepsilon) KoLuBi_m,
\]

\[
Bi_m > (1 - \varepsilon) PrKoLuBi_m + Bi_m Pn.
\]

The estimates demonstrating the convergence of the method are certainly above the accuracy of interest to the investigator, as a rule. We therefore require to solve the problem for 1, 2, 3, etc., straight lines, with the object of determining the relative discrepancy between the respective approximations. A tentative criterion of convergence of the method in a given region of variation of the arguments could be a monotonic decrease in these discrepancies as we increase the number of straight lines taken. A measure of the accuracy are the discrepancies, required especially in solution of non-linear problems, in which it is not possible to compare the approximate results with an exact analytical expression. It should be noted