ON THE STABILITY OF SECONDARY TAYLOR FLOW BETWEEN ROTATING CYLINDERS WITH A WIDE GAP

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In a general formulation, the problem of the stability of small perturbations in a homogeneous viscous fluid between rotating cylinders with a wide gap is investigated numerically.

The problem of the stability limits of a laminar viscous flow between rotating cylinders has been investigated by many authors starting with Taylor. The calculations were carried out initially under the assumption that the cylinder radii are approximately identical, which in turn permitted approximation of the velocity distribution in the steady flow either by a constant or by a linear function. A survey of the methods used to solve the problem in the above-mentioned approximation is presented in [1], where the majority of solutions refer to the case of cylinder rotation in one direction. The initial Taylor method, consisting in the expansion of the solution in orthogonal functions and in obtaining the characteristic equation in the form of an infinite determinant is also elucidated in [1].

The ideas of the Taylor method were extended and developed in [2], in which the curvature of the unperturbed flow velocity profile was taken into account and numerical results were obtained for the ratio between the cylinder radii $R_1/R_2 = 1/2$ in a small range of variation of their angular velocities.

A thorough analysis of experimental investigations on this problem is presented in [3], in which graphs are given of the stability for experiments with both a small and a wide gap between the cylinders. In the latter case, a comparison with existing theoretical computations is presented [2]. It is noted that the experiments were conducted in a significantly broader range of variation of the cylinder angular velocities than the calculations, and that there is no satisfactory mathematical description of the complete viscous problem for the case of cylinders rotating oppositely.

It is hence expedient to use finite-difference methods to solve this problem. The papers [4, 5] can be noted in this area.

A neutral curve is computed in [4] probably for the outer cylinder at rest and the ratio $R_1/R_2 = 1/2$ (these data are not presented in the paper).

A great volume of calculational work to determine the stability curves for a broad range of variation in the ratio $R_1/R_2$ is presented in [5]. The problem of determining the eigenvalues with respect to the Reynolds criterion is solved in this paper by using the Runge-Kutta method on a finite-difference matrix equation approximating the initial system of six first order equations for the neutral perturbations with undetermined parameters for part of the boundary conditions.

We used the method of numerically determining several of the first eigenvectors and their corresponding eigenvalues, elucidated in [6], to solve this problem, where the eigenvalue problem is posed relative to the exponent $\lambda$ in the exponential dependence of the solution on the time. The advantage of this method is that it permits obtaining a picture of the streamlines of the possible stable secondary flows directly, which are described by stream functions as components of the first eigenvector corresponding to the zero-th maximum eigenvalues of the problem.

1. Let us consider viscous fluid flow between cylinders of the radii $R_1, R_2$ rotating at the angular velocities $\Omega_1, \Omega_2$. The subscript 1 refers to the inner, and the subscript 2 to the outer cylinder. The
cylinders are not bounded along the z axis. The possible secondary Taylor flows are assumed periodic along the z axis with the wave parameter M and independent of the angle. Then the investigation of the stability of these flows in a linear approximation (see [1], Chapter 2) reduces to solving a system for the small perturbation amplitudes of the stream function \( \psi(\tau, r) \) and the velocity component \( v(\tau, r) \):

\[
\begin{align*}
\frac{1}{\Re} (L - M^2)^2 \psi - (L - M^2) \frac{\partial \psi}{\partial \tau} &= 2M\omega v, \\
\frac{1}{\Re} (L - M^2) v - \frac{\partial v}{\partial \tau} &= 2\alpha M \psi,
\end{align*}
\]

(1)

The system (1) is written in dimensionless quantities, where \( \Delta R = R_2 - R_1 \), \( \Delta R \Omega_1 \), and \( 1/\Omega_1 \) are taken as characteristic units for the length, velocity, and time, respectively.

From purely computational consideration (see [6]) it is expedient to replace the system (1) by a system of three second order equations by introducing an auxiliary function according to the relationship

\[
(L - M^2) \psi(\tau, r) = \psi(\tau, r).
\]

(2)

Using (2), let us write the system (1) in matrix form

\[
AX - B \frac{\partial X}{\partial \tau} = 0, \quad X = X(\psi, \psi, v),
\]

\[
A = \begin{bmatrix}
(L - M^2) & -1 & 0 \\
0 & \frac{1}{\Re} (L - M^2) & -2M\omega \\
-2M\alpha & 0 & \frac{1}{\Re} (L - M^2)
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

(3)

under the same boundary conditions.

If we separate variables in (3), we then obtain a problem to determine the eigenvalues in the following form

\[
AX - \lambda BX = 0, \quad \psi = \frac{d\psi}{dr} = v = 0 \text{ for } r = 1, \frac{R_2}{\Delta R},
\]

(4)

where the components of the vector \( X \) in this equation are functions only of the coordinate \( r \).

2. If the build-up method is used in the numerical determination of the eigenvectors of the problem (4) according to [6], the finite-difference problem approximating the system (3) can be written as