TEMPERATURE DISTRIBUTION IN LAMINAR FLOWS AT LARGE SHEAR STRESSES

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The energy equation is solved for the case of laminar flow of a compressible Newtonian liquid with allowance for heat of friction. Theoretical temperature profiles are presented for flow in channels with constant and linearly varying capillary wall temperatures. The theoretical values of the mean temperature are in good agreement with the experimental results.

When liquids move through capillaries under a pressure of several tens of atmospheres, the thermal effects of energy dissipation and compressibility become appreciable. These phenomena were reviewed in [1], where it was shown that the magnitudes of both effects are of the same order and that, in general, in internal flows the compressibility cannot be neglected.

In the first approximation, in flow through a thermally insulated capillary the average temperature of the liquid increases by about 1 °C for each 20–30 atm of pressure drop.

Under these conditions the simple forms of the equations of motion and energy, which hold true for flows in the region of small pressure drops, cease to be valid. Attention was first drawn to this by Hersey [2], who correctly attributed the experimentally observed deviations from the Poiseuille law to the effect of the heat of friction on the temperature field of the liquid and hence on its viscosity.

It is customary to distinguish two types of forced convection: a) adiabatic flow when there is no heat exchange with the ambient medium through the capillary walls; b) flow through an isothermal capillary at whose walls the initial (at the channel inlet) temperature of the liquid is kept constant.

Even recent publications confine themselves, with a few exceptions, to the problem of temperature distribution and give different variants of solutions of the energy equation. Some authors propose a solution of the energy equation that neglects the convection term [1–6]. Omitting the convective component limits the applicability of the solution to the region of the steady-state temperature profile. This region is important in connection with very long isothermal capillaries, in which equilibrium has been established between the dissipative component and the heat transmitted through the walls (in this case the temperature is a function of the radius only [1]).

Solutions for incompressible liquids in insulated or isothermal capillaries with allowance for convection are offered in [7–11].

Kudryashev and Golovin [12] have examined the effect of energy dissipation on heat transfer in a laminar channel flow of incompressible liquid at constant wall temperature.

Problems of the capillary flow of compressible liquids were first considered by Toor [1]. In [13] the same author discussed flow through an isothermal capillary with allowance for convection.

The solution proposed by Madejski [14] for a compressible liquid holds true only for a perfect gas [15]. A general mathematical model of laminar flow with allowance for heat of friction and expansion has been developed by Gee and Lyon [16]. These authors examined the flow of a non-Newtonian liquid in a heated or cooled channel with allowance for the variation of the physical parameters of the liquid. After publication of the work of Gee and Lyon the problem might have been considered solved in principle, if the nonlinear differential equations they obtained had been amenable to analytic solution. The authors used computers to find the temperature and velocity profiles as a function of the radius and length of the channel for several specific cases, obtaining values that agreed with the experimental results.

In this paper the problem is again examined by means of a simplified method based on the independence of the solutions of the equations of motion and energy. This is one of the approaches recommended by certain authors [17]; it may be regarded as the first stage of an iteration method.

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Formulation of the problem. The energy equation for a Newtonian liquid with constant thermal conductivity can be written in one of its alternative forms [18] as

$$\gamma c_p \frac{DT}{Dt} = \lambda \Psi^2 T + T \beta \frac{DP}{Dt} + \mu \Phi,$$  \hspace{1cm} (1)

where the coefficient of isobaric thermal expansion

$$\beta = \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_p.$$  \hspace{1cm} (2)

This is the general equation describing the temperature distribution for a real, dissipatively heated liquid in the absence of other internal heat sources (sinks).

Equation (1) has been solved on the basis of the following assumptions:

1. The physical parameters of the liquid are constant and do not depend on temperature and pressure (with the reservation mentioned in item 6).
2. The motion of the liquid has been stabilized.
3. The circumferential and radial components of velocity can be neglected. The velocity distribution in cross sections of the channel is parabolic.
Table 1  
Numerical Values of the Characteristic Constants $A_i$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$A_i$</th>
<th>$A_i'$</th>
<th>$A_i''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.3153</td>
<td>+0.146622</td>
<td>+0.3360</td>
</tr>
<tr>
<td>2</td>
<td>41.6903</td>
<td>+0.802476</td>
<td>+0.0719</td>
</tr>
<tr>
<td>3</td>
<td>113.921</td>
<td>+0.0579</td>
<td>+0.0330</td>
</tr>
<tr>
<td>4</td>
<td>215.241</td>
<td>-0.474897</td>
<td>-0.0330</td>
</tr>
</tbody>
</table>

4. The effect of external forces is neglected.
5. Heat conduction in the direction of the channel axis may be disregarded.
6. In accordance with [1] in a certain temperature interval it is possible to use the mean value of the product $T \beta = \varepsilon$.

With these assumptions Eq. (1) may be written in the simplified form [1]

$$\gamma \rho V \frac{\partial T}{\partial x} = \lambda \frac{1}{\gamma} \frac{\partial}{\partial r} \left( \frac{\partial T}{\partial r} \right) + \varepsilon V \frac{\partial \rho}{\partial x} + \mu \left( \frac{\partial V}{\partial r} \right)^2. \quad (3)$$

In Eq. (3) we have taken into account the appreciable influence of the compressibility of the liquid on the temperature distribution and have omitted the less important effect on the velocity and dissipation.

Under these conditions the equation of motion may be written in the form

$$\frac{dp}{dx} = \frac{1}{r} \frac{d}{dr} \left( \mu r \frac{dV}{dr} \right). \quad (4)$$

With the above assumptions Eqs. (3) and (4) are independent. Replacing the pressure gradient with the average pressure drop along the capillary $dp/dx = (-P/\delta)$, on the basis of Eqs. (3) and (4) we can write

$$\gamma \rho V \left( \frac{p}{x} \right)\frac{R^2}{4 \mu} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \frac{\partial T}{\partial x} = \lambda \frac{1}{\gamma} \frac{\partial}{\partial r} \left( \frac{\partial T}{\partial r} \right) + \varepsilon \frac{R^2}{4 \mu} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \frac{p}{x} \frac{\partial V}{\partial r}. \quad (5)$$

After rearranging and introducing the relative radius $\rho = r/R$, together with Brinkman's relative length [7]

$$z = \frac{4 \lambda \mu x}{\gamma \rho R^4 (p/x)} = \frac{1}{\rho} \left( \frac{p}{x} \right) = \frac{\pi}{2} \left( \text{Gm} \right)^{-1} \quad (6)$$

and the relative temperature

$$\tau = \frac{16 \lambda \mu T}{R^4 (p/x)^2} = 2 \pi \left( \text{Gm} \right)^{-1} \left( \frac{\varepsilon \gamma T}{P} \right), \quad (7)$$

we obtain the final form of the differential equation of temperature distribution for laminar flow of a Newtonian compressible liquid with allowance for the heat of friction:

$$(1 - \rho^2) \frac{\partial \tau}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \tau}{\partial \rho} \right) + 4 \left( 1 + \varepsilon \right) \rho^2 - \varepsilon. \quad (8)$$

In this paper Eq. (8) has been solved for a Newtonian compressible liquid flowing in a capillary with a constant arbitrary wall temperature and in a capillary with a wall temperature that varies linearly with length.

Solution of the equations. A. Wall temperature constant. This case is characterized by the following boundary conditions:

$$\tau (\rho, z), \tau (1, z) = \tau_w = \text{const}; \quad 0 \leq \rho \leq 1, \quad \tau (\rho, 0) = \tau_0; \quad 0 \leq z \leq \infty, \quad \left( \frac{\partial \tau}{\partial \rho} \right)_{\rho=0} = 0. \quad (9)$$

Introducing the new variable

$$\theta = (\tau - \tau_0) / (\tau_0 - \tau_w), \quad (10)$$

we can write Eqs. (9) in the form

$$\left( \frac{1 - \rho^2}{\rho} \right) \frac{\partial \theta}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \theta}{\partial \rho} \right) + \frac{4 \left( 1 + \varepsilon \right) \rho^2}{\left( \tau_0 - \tau_w \right)} - \frac{4 \varepsilon}{\left( \tau_0 - \tau_w \right)}. \quad (11)$$

In the new notation the boundary conditions are

$$\theta (\rho, z), \theta (1, z) = 0; \quad \theta (\rho, 0) = 1; \quad \left( \frac{\partial \theta}{\partial \rho} \right) = 0. \quad (12)$$

We will find [7, 9] the solution of Eq. (12) in the form of a sum of two functions

$$\theta = \theta_1 + \theta_2, \quad (13)$$

where $\theta_1$ is the solution for the case $\partial \theta / \partial z = 0$, and $\theta_2$ is an auxiliary function required to obtain the general solution.

The first term is found on the assumption that the convection term may be neglected, $\partial \theta / \partial z = 0$. We then obtain

$$\theta_1 = \frac{1}{4} \left( 1 + \varepsilon \right) \left( 1 - \rho^2 \right) - \frac{4 \varepsilon}{\left( \tau_0 - \tau_w \right)}, \quad (14)$$

Substituting (14) in (13) and then in (11), we reduce the problem to the classical case of heat transfer in a laminar flow:

$$\left( 1 - \rho^2 \right) \frac{\partial \theta_2}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \theta_2}{\partial \rho} \right). \quad (15)$$

Representing the function $\theta_2$ in the form of a product

$$\theta_2 (\rho, z) = \Phi (\rho) \psi (z),$$