TEMPERATURE FIELD IN PLATES AND FLAT SHELLS WITH INTERNAL HEAT SOURCES

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Solutions of the problem of heat conduction with boundary conditions of the first, second, and third kinds are obtained for an infinite plate exposed to one of the following influences: instantaneous point heat source, initial temperature concentrated at a point, or instantaneous point action of a medium at its surface.

Green's functions of the problem of heat conduction for plates and flat shells. Let a homogeneous isotropic plate be heated by internal heat sources. The temperature at any point of the plate is given by the equation

$$\frac{\partial \Theta}{\partial \tau} - \Delta \Theta = \psi_0 (\alpha, \beta, \gamma, \tau)$$

and the boundary conditions

$$-a_1 \frac{\partial \Theta}{\partial \gamma} + b_1 \Theta = \psi_1 (\alpha, \beta, \tau) \text{ at } \gamma = 0;$$
$$a_2 \frac{\partial \Theta}{\partial \gamma} + b_2 \Theta = \psi_2 (\alpha, \beta, \tau) \text{ at } \gamma = 1;$$
$$\Theta (\alpha, \beta, \gamma, \tau) = \psi_3 (\alpha, \beta, \gamma) \text{ at } \tau = 0.$$

Equation (1) is also applicable to thin-walled flat shells [2].

We will find the distribution of temperature \( \Theta^* \) in the plate when the functions \( \psi_i \) are given in the form

$$\psi_0 (\alpha, \beta, \gamma, \tau) = \frac{\delta (\alpha - \alpha_0)}{\alpha_0} \delta (\beta - \beta_0) \delta (\gamma - \gamma_0) \delta (\tau);$$
$$\psi_1 (\alpha, \beta, \tau) = \frac{\delta (\alpha - \alpha_1)}{\alpha_1} \delta (\beta - \beta_1) \delta (\tau);$$
$$\psi_2 (\alpha, \beta, \gamma) = \frac{\delta (\alpha - \alpha_2)}{\alpha_2} \delta (\beta - \beta_2) \delta (\gamma);$$
$$\psi_3 (\alpha, \beta, \gamma) = \frac{\delta (\alpha - \alpha_3)}{\alpha_3} \delta (\beta - \beta_3) \delta (\gamma - \gamma_3).$$

We represent \( \delta (\beta - \beta') \) in series form:

$$\delta (\beta - \beta') = \frac{1}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \cos m (\beta - \beta'),$$

and, correspondingly, write

$$\Theta^* = \frac{1}{\pi} \sum_{m=0}^{\infty} \varepsilon_m \Theta^*_m \cos m (\beta - \beta').$$

The coefficients \( \Theta^*_m \) satisfy the equation

$$\frac{\partial \Theta^*_m}{\partial \tau} - \frac{1}{\alpha} \frac{\partial}{\partial \alpha} \left( \frac{\alpha}{\partial \alpha} \frac{\partial \Theta^*_m}{\partial \alpha} \right) + \frac{m^2}{\alpha^2} \frac{\partial^2 \Theta^*_m}{\partial \gamma^2} = \frac{\delta (\alpha - \alpha_0)}{\alpha_0} \delta (\gamma - \gamma_0) \delta (\tau),$$

and the boundary conditions

$$-a_1 \frac{\partial \Theta^*_m}{\partial \gamma} + b_1 \Theta^*_m = \frac{\delta (\alpha - \alpha_1)}{\alpha_1} \delta (\tau) \text{ at } \gamma = 0;$$
$$a_2 \frac{\partial \Theta^*_m}{\partial \gamma} + b_2 \Theta^*_m = \frac{\delta (\alpha - \alpha_2)}{\alpha_2} \delta (\tau) \text{ at } \gamma = 1;$$
$$\Theta^*_m (\alpha, \beta, \gamma) = \frac{\delta (\alpha - \alpha_3)}{\alpha_3} \delta (\gamma - \gamma_3) \text{ at } \tau = 0.$$

To solve Eq. (4) we use Doetsch integral transforms [3] with respect to the variable \( \gamma \) and Hankel transforms with respect to \( \alpha \). We denote the double transform of \( \Theta^*_m \) by \( \Theta_{mn} \), i.e.,

$$\Theta_{mn} = \int_0^\infty \int_0^\infty \Theta^*_m (\alpha, \gamma) d\alpha d\gamma,$$

where

$$Z_n (\gamma) = A_n \cos \mu_\gamma + B_n \sin \mu_\gamma$$

are solutions of the problem

$$Z_n ^{\prime\prime} (\gamma) + \mu^2_n Z_n (\gamma) = 0$$

with the following boundary conditions:

$$-a_1 Z_n ^{\prime} (0) + b_1 Z_n (0) = 0;$$
$$a_2 Z_n ^{\prime} (1) + b_2 Z_n (1) = 0.$$

The determinant of system (7), equated to zero, gives the characteristic equation for finding the eigenvalues \( \mu_n^2 \):

$$\tan \mu_n = \mu_n (a_1 b_1 + b_2 a_2)/(a_2 b_2 + b_1 a_1).$$

Investigation shows that the problem does not have zero eigenvalues \( \mu_n^2 = 0 \), except for the case of \( b_1 = b_2 = 0 \).

We find the coefficients \( A_n \) and \( B_n \) from (7) and also from the normalization condition

$$\int_0^\infty Z_n ^{\prime} (\gamma) d\gamma = 1.$$

After evaluation we obtain

$$B_n ^2 = \frac{2b_2^2 (a_2 \mu_n^2 + b_2^2)}{(a_1^2 \mu_n^2 + b_1^2) (a_2^2 \mu_n^2 + b_2^2) + (a_1 b_2 \mu_n^2 + b_1 a_2) (a_2 b_1 + b_2 a_1)};$$

$$A_n = \frac{a_1 \mu_n}{b_1} B_n$$

for \( n \geq 1 \);

$$B_0 = 0;$$

$$A_0 = \begin{cases} 1 & \text{at } b_1 = b_2 = 0; \\ 0 & \text{otherwise}. \end{cases}$$
Multiplying expression (4) and the last of Eqs. (5) by the kernel of the transform and then integrating

\[ \int_0^1 \frac{d^2 G_m(u a)}{d \gamma^2} Z_n(\gamma_1) d \gamma = \sum_{i=0}^{2} c_i(\gamma_1) Z_N(\gamma_1) J_m(u a_1) \delta(\tau), \]

\[ \Theta_{mn}(0) = J_m(u a_0) Z_n(\gamma_1). \]  

Here, we have taken into account the fact that

\[ \int_0^1 \frac{d^2 G_m}{d \gamma^2} Z_n(\gamma) d \gamma = \sum_{i=0}^{2} c_i(\gamma_1) Z_N(\gamma_1) \frac{\delta(x - a_i)}{a_i} \delta(\tau) - \mu_2^2 \int_0^1 \theta^*_m(\gamma) d \gamma. \]

The solution of Eq. (9) will be

\[ \Theta_{mn} = \sum_{i=0}^{3} c_i(\gamma_1) Z_N(\gamma_1) J_m(u a_1) \exp \left(-a_i^2 \tau^2 \right), \]

where

\[ c_i(\gamma_1) = \begin{cases} \frac{1}{a_i} & \text{at } a_i \neq 0; \\ (-1)^{i+1} \frac{d}{d \gamma_i} & \text{at } a_i = 0; \end{cases} \]

\[ a_0 = a_2 = 1; \quad \gamma_1 = 0; \quad \gamma_2 = 1. \]

The inverse transformation gives

\[ \theta^*_m = \sum_{i=0}^{3} c_i(\gamma_1) Z_N(\gamma_1) Z_n(\gamma) \exp \left(-\mu_2^2 \tau^2 \right) \times \int_0^1 u J_m(u a_1) J_m(u a) \exp \left(-a_i^2 \tau^2 \right) du. \]

Using the formula for the addition of cylindrical functions, we sum \( \delta^* \), in accordance with (3), and after integration obtain

\[ \theta^* = \sum_{i=0}^{3} G_i(a, a_1, a_2, \beta', \gamma, \gamma_1, \tau), \]

where

\[ G_i(a, a_1, a_2, \beta', \gamma, \gamma_1, \tau) = \frac{1}{4 \pi t} \exp \left(-\frac{R_i^2}{4 \tau} \right) \times \sum_{n=0}^{\infty} c_i(\gamma_1) Z_N(\gamma) Z_n(\gamma) \exp \left(-\mu_2^2 \tau \right); \]

\[ R_i = \sqrt{a_1^2 + a_2^2 - 2a_1a_2 \cos(\beta - \beta')}. \]  

Results (11) can be used in an arbitrary coordinate system on the surface of the plate (\( \gamma = 0 \)), if it is kept in mind that \( R \) is the distance between the center of action and the instantaneous point. In particular, in a Cartesian coordinate system \( R_i = [(x - x_1)^2 + (y - y_1)^2]^{1/2} \).

Each individual solution \( G_i \) gives the temperature field in the plate due to just one factor: concentrated instantaneous heat source (i = 0), concentrated instantaneous action of the medium at the surfaces of the plate (i = 1, 2), concentrated action of the initial temperature (i = 3).

Solutions (11) may be regarded as Green's functions of the problem (1), (2), which can be used to solve that problem for an arbitrary distribution of heat source intensity and arbitrary boundary conditions given by the functions \( \psi_i \):

\[ \theta(a, \beta, \gamma, \tau) = \int_0^1 a_0 d a_0 \int_0^1 d \beta' \int_0^1 d \psi \times \]

\[ \times \int_0^1 \hat{G}_i(a, \alpha, \beta, \beta', \gamma, \gamma_1, \tau) \times \psi_i(\alpha, \beta', \tau - \tau') d \tau' \]  

\[ + \int_0^1 \int_0^1 \int_0^1 d \beta' \int_0^1 \hat{G}_i(a, \alpha, \beta, \beta', \gamma, \gamma_1, \tau) \times \]

\[ \times \psi_i(\alpha, \beta', \tau - \tau') d \tau' \]  

\[ \times \int_0^1 \hat{G}_i(a, \alpha, \beta, \beta', \gamma, \gamma_2, \tau) \times \psi_i(\alpha, \beta', \gamma_2, \tau) d \tau. \]  

We note that result (12) can be obtained mathematically from Eq. (1) for homogeneous boundary conditions (2) if the heat source intensity is given by a specific function, i.e.,

\[ \frac{\partial \theta}{\partial \gamma} - \Delta \theta = \psi(a, \beta, \gamma, \tau) + \]

\[ + \sum_{i=1}^{2} c_i(\gamma) \psi_i(a, \beta, \gamma) \delta(\gamma - \gamma_i) + \delta(\tau) \psi(a, \beta, \gamma); \]

\[ - a_1 \frac{\partial \theta}{\partial \gamma} + b_1 \theta = 0 \text{ at } \gamma = 0; \]

\[ a_2 \frac{\partial \theta}{\partial \gamma} + b_2 \theta = 0 \text{ at } \gamma = 1; \]

\[ \theta(a, \beta, \gamma, \tau) = 0 \text{ at } \tau = 0. \]

Consequently, problem (1), (2) reduces to the problem with homogeneous boundary conditions. This result can be formulated as follows: the action of the medium on the plate is equivalent to heat sources (if \( a_1 = 0 \)) or dipoles (if \( a_1 = 0 \)) distributed over the surfaces of the plate, and the action of the initial temperature is equivalent to instantaneous heat sources acting at the initial instant inside the plate. This conclusion was reached in [4] in the case of the one-dimensional problem with boundary conditions of the first kind (\( a_1 = 0 \)).

Our method employing Doetsch integral transforms gives the solution in the form of series in eigenfunctions, these series converging absolutely and uniformly in all closed regions on the interval \((0, 1)\).

The constants \( a_1, a_2, b_1, b_2 \) in boundary conditions (2) may take arbitrary nonnegative values, i.e., ex-